MATH215A, HW2

Solutions

10/18/11

Problem 1. *3.1*

$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$. $\sin \pi z = 0 \Leftrightarrow e^{i\pi z} = e^{-i\pi z} \Leftrightarrow e^{2i\pi z} = 1 \Leftrightarrow z \in Z$. Zero of order 1 is equivalent to order 1 of the pole of $1/\sin \pi z$. The second we get by seeing that $\lim_{z\to 0} \frac{z}{sin\pi z} = 1/\pi$ (if the order was not 1 then that limit would not converge). The residue is $1/\pi$ (the function is clearly periodic so the residues at the even integers are all $1/\pi$ and by the fact that it's odd we get that the residues at odd integers are $-1/\pi$).

Problem 2. *3.14*

Look at $f(1/z)$. If it has an essential singularity at 0, then pick any $z_0 \neq 0$. Now we know that the range of f is dense as $z \to 0$. We also know that the image of f in some small ball around z_0 contains a ball around $f(z_0)$. But this means that the image of f around this ball intersects the image of f in any arbitrarily small ball around 0 (because of the denseness). Thus, f cannot be injective. So the singularity at 0 is not essential, so $f(1/z)$ is some polynomial of $1/z$, so f is some polynomial of z. If its degree is more than 1 it is not injective (fundamental theorem of algebra), so the degree of f is 1.

Problem 3. *3.15*

a) Let $g(z) = f(1/z)$. Then we have that $\sup_{|z|=R} |g(z)| \leq A/R^k + B \Rightarrow \sup_{|z|=R} |z^k g(z)| \leq A + B * R^k$, so $z^k f(1/z)$ is bounded around 0, so it has a non-essential singularity at 0, so f has a non-essential singularity at ∞, so it is a polynomial. The fact that the degree is $\leq k$ follows directly, since it is trivial that for large enough $z \in R$ $P(z) > 0$ for any polynomial with positive first coefficient.

b) We can do a FLT to make the circle into the line $Re(z) = 1/2$ ($h(z) = 1/(z+1)$). Then the inner circle becomes the right half-plane. Then we can rotate this s.t. we get the upper half-plane with removed the $(0,1)$ interval. But we have a theorem (Schwarz reflection principle) that says that we can extend f on the lower half-plane as well. But then we have a set on which f is constantly 0, so f is constantly 0.

c) Look at $p(z) = (z - w_1)...(z - w_k)$. We have $|p(0)| = 1$. But the maximum principle we have that there is a point on the circle for which $|p(z)| \geq 1$, which is what we wanted. Since $|p(z)|$ is continuous on the circle and $p(w_1) = 0$ we have by the intermediate value theorem that there is a z for which $|p(z)| = 1$.

d) Look at $e^{f(z)}$. This is also entire and $|e^{f(z)}| = |e^{Re(f(z))}|$, so it is bounded. So it is constant. So $f(z)$ is constant.

Problem 4. *3.16*

a) f has a unique zero at $z = 0$, so it is bounded from below on the circle. g is analytic so it is bounded from above on the circle (compact set). Thus, there is an $\epsilon > 0$ s.t. $|f| > \epsilon|g|$ on the circle, so the number of zeros of f and $f + \epsilon g$ in the disk is the same.

b) Suppose it is not. Then there exists a $x \in (0, \epsilon)$ s.t. there exists a sequence $x_1, ...$ converging to x s.t. the sequence $z_{x_1}, ...$ doesn't converge to z_x. But the disk is a compact set, so this sequence has a convergent subseqence converging to, say, z_0. WLOG let $\lim z_{x_i} = z_0$. But then, if $|(f + xg)(z_0)| = k > 0$. In particular, there is some ball around z_0 s.t. $|f + xg| > k/2$ in this ball. So then if we pick a close enough x_i to x (with i big enough s.t. z_{x_i} is in this ball), using the fact that g is bounded from above, we will get that $|(f + x_i g)(z_{x_i})| > 0$, which is a contradiction. So the function is continuous.

Problem 5. *3.17*

a) Suppose it does not contain a root. Then $1/f(z)$ is holomorphic. But then by the maximum principle (since again $|1/f(z)| = 1$ on the circle), we have that $|f(z)| = 1$ in the entire disk. But an analytic non-constant function cannot have constant absolute value (since it is an open map), so we have a contradiction. So we have a root of $f(z) = 0$. But now Rouche's theorem tells us that $f(z) = w$ has a root for every w in the unit disk (since on the circle $|f| = 1 > |w|$).

b) The exact same argument works - again we look at $1/f(z)$ and since we already have a point for which $|f(z)| < 1 \Rightarrow |1/f(z)| > 1$, this cannot be analytic in the unit disk. Thus, f has a zero, after which we can again apply Rouche's Theorem.

Problem 6. *3.2*

$1/(x^4+1) = \frac{1}{2i}(1/(x^2-i) - 1/(x^2+i)) = \frac{1}{2i}(\frac{1}{2a}(1/(x-a) - 1/(x+a)) - \frac{1}{2b}(1/(x-b) - 1/(x+b)))$, where $a = e^{i\pi/4}$, $b = e^{3i\pi/4}$. Now, integrating along the contour consisting of a big upper semicircle of radius R and $[-R, R]$ and using the fact that the integral along the semicircle converges to 0 (trivial - length grows linearly, the integrand grows like R^{-4}) we get that the integral equals $\frac{2\pi i}{2i}(\frac{1}{2a} + \frac{1}{2b}) = \pi/2(e^{i\pi/4} + e^{-i\pi/4}) = \pi\sqrt{2}/2$ (we only care about the poles in the upper half-plane). The poles are $e^{i\pi/4}, e^{3i\pi/4, e^{-3i\pi/4}}, e^{-i\pi/4}$.

Problem 7. *3.3*

$cosx = \frac{e^{ix}+e^{-ix}}{2}$. changing $x \to -x$ we see that we can just integrate $e^{ix}/(x^2+a^2)$ and we'll get the same answer. Again, we use the same semicircle and part of the real line. The only pole is $x = ia$, it has order 1 and the residue at it is $\lim_{x\to ia} \frac{e^{ix}}{x^2+a^2}(x - ia) = \frac{e^{-a}}{2ia}$, which multiplied by $2\pi i$ gives the answer.

Problem 8. *3.4*

$x/(x^2+a^2) = x/2ia(1/(x-ia) - 1/(x+ia)) = 1/2ia(ia/(x-ia) + ia/(x+ia)) = (1/(x-ia) + 1/(x+ia))/2$. So we care about $\sin(x)(1/(x-ia) + 1/(x+ia))/2$. Its residue at $x = ia$ is $\sin(ia)/2 = (e^{-a} - e^{a})/4i$. ?

Problem 9. *3.6*

Integrate along the same contour. The only pole is at $z = i$ and is of order $n+1$. So we look at $1/n!(\frac{d}{dn}\frac{1}{(x+i)^{n+1}})(i) = 1/n!(n+1)...(2n)/2^{2n} = (2n)!/(2^{2n}(n!)^2)/2i$, which multiplied by $2\pi i$ gives the answer.

Problem 10. *3.9*

The left and right parts are equal because of symmetry ($sinx = -sin(x+\pi)$). The upper part should equal $\log(2)$, but I don't know why.