Problem 1. 3.1

\[ \sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}. \quad \sin \pi z = 0 \iff e^{i\pi z} = e^{-i\pi z} \iff e^{2i\pi z} = 1 \iff z \in \mathbb{Z}. \]

Zero of order 1 is equivalent to order 1 of the pole of \( \frac{1}{\sin \pi z} \). The second we get by seeing that \( \lim_{z \to 0} \frac{z}{\sin \pi z} = 1/\pi \) (if the order was not 1 then that limit would not converge). The residue is \( 1/\pi \) (the function is clearly periodic so the residues at the even integers are all \( 1/\pi \) and by the fact that it’s odd we get that the residues at odd integers are \(-1/\pi\)).

Problem 2. 3.14

Look at \( f(1/z) \). If it has an essential singularity at 0, then pick any \( z_0 \neq 0 \). Now we know that the range of \( f \) is dense as \( z \to 0 \). We also know that the image of \( f \) in some small ball around \( z_0 \) contains a ball around \( f(z_0) \). But this means that the image of \( f \) around this ball intersects the image of \( f \) in any arbitrarily small ball around 0 (because of the denseness). Thus, \( f \) cannot be injective. So the singularity at 0 is not essential, so \( f(1/z) \) is some polynomial of \( 1/z \), so \( f \) is some polynomial of \( z \). If its degree is more than 1 it is not injective (fundamental theorem of algebra), so the degree of \( f \) is 1.

Problem 3. 3.15

a) Let \( g(z) = f(1/z) \). Then we have that \( \sup_{|z|=R} |g(z)| \leq A/R^k + B \Rightarrow \sup_{|z|=R} |z^k g(z)| \leq A + B \ast R^k \), so \( z^k f(1/z) \) is bounded around 0, so it has a non-essential singularity at 0, so \( f \) has a non-essential singularity at \( \infty \), so it is a polynomial. The fact that the degree is \( \leq k \) follows directly, since it is trivial that for large enough \( z \in \mathbb{R} \) \( P(z) > 0 \) for any polynomial with positive first coefficient.

b) We can do a FLT to make the circle into the line \( Re(z) = 1/2 \) (\( h(z) = 1/(z+1) \)). Then the inner circle becomes the right half-plane. Then we can
rotate this s.t. we get the upper half-plane with removed the (0, 1) interval. But we have a theorem (Schwarz reflection principle) that says that we can extend \( f \) on the lower half-plane as well. But then we have a set on which \( f \) is constantly 0, so \( f \) is constantly 0.

c) Look at \( p(z) = (z - w_1)(z - w_k) \). We have \(|p(0)| = 1\). But the maximum principle we have that there is a point on the circle for which \(|p(z)| \geq 1\), which is what we wanted. Since \(|p(z)|\) is continuous on the circle and \( p(w_1) = 0 \) we have by the intermediate value theorem that there is a \( z \) for which \(|p(z)| = 1\).

d) Look at \( e^{f(z)} \). This is also entire and \(|e^{f(z)}| = |e^{Re(f(z))}|\), so it is bounded. So it is constant. So \( f(z) \) is constant.

**Problem 4. 3.16**

a) \( f \) has a unique zero at \( z = 0 \), so it is bounded from below on the circle. \( g \) is analytic so it is bounded from above on the circle (compact set).

b) Suppose it is not. Then there exists a \( x \in (0, \epsilon) \) s.t. there exists a sequence \( x_1, \ldots \) converging to \( x \) s.t. the sequence \( z_{x_1}, \ldots \) doesn’t converge to \( z_x \). But the disk is a compact set, so this sequence has a convergent subsequence converging to, say, \( z_0 \). WLOG let \( \lim z_{x_i} = z_0 \). But then, if \(|(f + xg)(z_0)| = k > 0\). In particular, there is some ball around \( z_0 \) s.t. \(|f + xg| > k/2\) in this ball. So then if we pick a close enough \( x_i \) to \( x \) (with \( i \) big enough s.t. \( z_{x_i} \) is in this ball), using the fact that \( g \) is bounded from above, we will get that \(|(f + x_i g)(z_{x_i})| > 0\), which is a contradiction. So the function is continuous.

**Problem 5. 3.17**

a) Suppose it does not contain a root. Then \( 1/f(z) \) is holomorphic. But then by the maximum principle (since again \(|1/f(z)| = 1\) on the circle), we have that \(|f(z)| = 1\) in the entire disk. But an analytic non-constant function cannot have constant absolute value (since it is an open map), so we have a contradiction. So we have a root of \( f(z) = 0 \). But now Rouche’s theorem tells us that \( f(z) = w \) has a root for every \( w \) in the unit disk (since on the circle \(|f| = 1 > |w|\)).

b) The exact same argument works - again we look at \( 1/f(z) \) and since we already have a point for which \(|f(z)| < 1 \Rightarrow |1/f(z)| > 1\), this cannot be analytic in the unit disk. Thus, \( f \) has a zero, after which we can again apply Rouche’s Theorem.
Problem 6. 3.2

\[\frac{1}{(x^4 + 1)} = \frac{1}{2\pi i} \left( \frac{1}{(x^2 - i)} - \frac{1}{(x^2 + i)} \right) = \frac{1}{2\pi i} \left( \frac{1}{(x - a)} - \frac{1}{(x + a)} \right) - \frac{1}{2\pi i} \left( \frac{1}{(x - b)} - \frac{1}{(x + b)} \right), \quad a = e^{i\pi/4}, \quad b = e^{3i\pi/4}. \]

Now, integrating along the contour consisting of a big upper semicircle of radius \( R \) and \([-R, R]\) and using the fact that the integral along the semicircle converges to 0 (trivial - length grows linearly, the integrand grows like \( R^{-4} \)) we get that the integral equals \( 2\pi i \cdot \frac{1}{2} \left( e^{i\pi/4} + e^{-i\pi/4} \right) = \pi \sqrt{2} / 2 \) (we only care about the poles in the upper half-plane). The poles are \( e^{i\pi/4}, e^{3i\pi/4}, e^{-3i\pi/4}, e^{-i\pi/4}. \)

Problem 7. 3.3

\[\cos x = \frac{e^{ix} + e^{-ix}}{2}. \quad \text{changing \( x \to -x \) we see that we can just integrate} \]
\[e^{ix} / (x^2 + a^2) \text{ and we'll get the same answer. Again, we use the same semi-}
\[\text{circle and part of the real line. The only pole is} \quad x = ia, \quad \text{it has order 1 and the residue at it is} \]
\[\lim_{x \to ia} \frac{e^{ix}}{x^2 + a^2} = \frac{e^{-a}}{2ia}, \quad \text{which multiplied by} \quad 2\pi i \quad \text{gives the answer.} \]

Problem 8. 3.4

\[x / (x^2 + a^2) = x / 2ia(1/(x - ia) - 1/(x + ia)) = 1/2ia(ia/(x - ia) + ia/(x + i a)) = (1/(x - ia) + 1/(x + ia)) / 2. \quad \text{So we care about} \]
\[\sin(x)(1/(x - ia) + 1/(x + ia)) / 2. \quad \text{Its residue at} \quad x = ia \quad \text{is} \quad \sin(ia)/2 = (e^{-a} - e^a)/4i. \]

Problem 9. 3.6

Integrate along the same contour. The only pole is at \( z = i \) and is of order \( n + 1 \). So we look at \( 1/n! \left( \frac{d}{dz} \frac{1}{(z + i)^{n+1}} \right) (i) = 1/n! (n + 1) \ldots (2n) / 2^{2n} = (2n)! / (2^{2n} (n!)^2) / 2i, \) which multiplied by \( 2\pi i \) gives the answer.

Problem 10. 3.9

The left and right parts are equal because of symmetry \( (\sin x = -\sin(x + \pi)) \). The upper part should equal \( \log(2) \), but I don’t know why.