MATH215A, Homework 1

Solutions

10/11/11

Problem 1. *1.6*

(a) C_x is open: Let $y \in C_x \subseteq \Omega$. Then $\exists \epsilon > 0$ s.t. $D_\epsilon(y) \subseteq \Omega$, so for any $z \in D_\epsilon(y)$ we have $z \in C_x$ since we can join x and z by first going from x to y and then from y to z (say, by following the straight line from y to z). C_x is connected: Suppose it was not. Then we can split it into two open sets A and B. Suppose $x \in A$ and $y \in B$ and look at the curve $f : [0,1] \to C$ connecting x and y. Let $t_0 = \inf_t(f(t) \in B)$ (it exists because $y \in B$ and is > 0 because $x \notin B$). If $f(t_0) \in B$ then openness of B and continuity of f shows that $f(t_0 - \epsilon) \in B$ for some $\epsilon > 0$ which contradicts minimality of t_0. If $f(t_0) \notin B$, then $f(t_0) \in A$ and again continuity of f implies that $f(t_0 + \epsilon) \in A$ for any ϵ small, which contradicts the definition of t_0 (otherwise stated: path connected componenets are subsets of connected components).
The equivalence relation:
(i) $z \in C_z$ trivially, since z is connected to itself via the empty path (or, if we don't like empty paths, via a sufficiently small circle which z lies on, which we can find by using the openness of C_z).
(ii) Let $w \in C_z$. Then there is a path γ from w to z. But then $-\gamma$ is a path from z to w, son $z \in C_w$.
(iii) Let $a \in C_b$, $b \in C_c$. Then a path from a to c we can get by concatenating the paths from a to b and from b to c, so $a \in C_c$.

(b) Any connected component is open, so it contains a rational point (since these are dense in R^2). Thus, by picking any rational point for a given component we define an injective map from the set of connected components to the set of rational numbers, so the set of connected components is at most countable.

(c) A compact set is bounded. Put it in some large circle, e.g. $D_R(0)$. Then any two points that are outside the circle are in the same connected component (a path from x to y we can define by e.g. following the lines from x to 0 and from y to 0 until they intersect $C_R(0)$ and connecting the two points on the circle by just going around the circle). Any unbounded component contains at least one point outside $D_R(0)$, so any unbounded component coincides with the component of the points outside $D_R(0)$. Thus, there is only one unbounded component.

Problem 2. *1.7*

(a) $|\frac{w-z}{1-\overline{w}*z)}| \leq 1 \Leftrightarrow (w-z)*\overline{(w-z)} \leq (1-\overline{w}*z)*\overline{(1-\overline{w}*z)} \Leftrightarrow$ $(1-w*\overline{w})*(1-z*\overline{z}) \geq 0$ (the last is a simple manipulation), which is true since $x * overlinex = |x|^2 < 1$ for $|x| < 1$. Equality for $|z| = 1$ or $|w| = 1$ follows directly.

(b) F maps the unit disk to itself because of the inequality we have just proven. It is holomorphic as a quotient of two holomorphic functions (the denominator is non-zero because $|\overline{w} * z| = |w| * |z| < |z| \leq 1$). By direct calculation $F(0) = \frac{w-0}{1-\overline{w}*0} = w, F(w) = \frac{w-w}{1-\overline{w}*w} = 0$. $|F(z) = 1$ for $z = 1$ because of the equality case of (a). $F(F(z)) = \frac{w-F(z)}{1-\overline{w}F(z)} = z$ by direct calculation. Thus, F is bijective.

Problem 3. *1.9*

$\frac{du}{dr} = \frac{du}{dx}\frac{dx}{dr} + \frac{du}{dy}\frac{dy}{dr} = \frac{du}{dx}\cos\theta + \frac{du}{dy}\sin\theta$, $\frac{dv}{dr} = \frac{dv}{dx}\frac{dx}{dr} + \frac{dv}{dy}\frac{dy}{dr} = \frac{dv}{dx}\cos\theta + \frac{dv}{dy}\sin\theta$, $\frac{du}{d\theta} = \frac{du}{dx}\frac{dx}{d\theta} + \frac{du}{dy}\frac{dy}{d\theta} = -r*\frac{du}{dx}\sin\theta + r*\frac{du}{dy}\cos\theta$, $\frac{dv}{d\theta} = \frac{dv}{dx}\frac{dx}{d\theta} + \frac{dv}{dy}\frac{dy}{d\theta} = -r*\frac{dv}{dx}\sin\theta + r*\frac{dv}{dy}\cos\theta$. Now a direct application of the Cauchy-Riemann equations shows the validity of the statements we are required to prove. For $f(z) = \log z$ we have $\frac{du}{dr} = \frac{1}{r}$, $\frac{dv}{d\theta} = 1$, $\frac{du}{d\theta} = 0 = \frac{dv}{dr}$, so the equations are satisfied, so the function is holomorphic.

Problem 4. *1.10*

Let $f(x+i*y) = u+i*v$. Then $\frac{df}{dz} = 1/2*(u_x - i*u_y + i*v_x + v_y) = 1/2*((u_x+v_y)+i*(v_x-u_y))$, $\frac{df}{d\overline{z}} = 1/2*(u_x + i*u_y + i*v_x - v_y) = 1/2*((u_x-v_y)+i*(v_x+u_y))$. Applying $\frac{d}{d\overline{z}}$ to the first gives $1/4*(u_{xx} + v_{yx} - v_{xy} + u_{yy} + i*(v_{xx} - u_{yx} + u_{yx} + v_{yy})) = 1/4*(u_{xx}+u_{yy}+i*(v_{xx}+v_{yy}))$. Applying $\frac{d}{dz}$ to the second gives $1/4*(u_{xx} - v_{yx} + v_{xy} + u_{yy} + i*(v_{xx} + u_{yx} - u_{yx} + v_{yy})) = 1/4*(u_{xx}+u_{yy}+i*(v_{xx}+v_{yy}))$. So we need to show $u_{xx}+u_{yy}+i*(v_{xx}+v_{yy}) = \frac{d^2f}{dx^2} + \frac{d^2f}{dy^2}$ which is true by just expanding the RHS by definition.

Problem 5. *1.11*

We know that if f is holomorphic, then $\frac{df}{d\bar{z}} = 0$, which according to the previous exercise means that the Laplacian is zero. Alternatively, we can get the same result by just differentiating the Cauchy-Riemann equations.

Problem 6.

(d) $((3n)!/(n!)^3)^{1/n} = (1+o(1))^{1/n} * (((3n)^{3n+1/2}e^{-3n})/(n^{3n+3/2}e^{-3n}))^{1/n} = (1+o(1)) * 3^3 * n^{-1/n} = (1+o(1)) * 27$, so the radius is 27 ($n^{1/n} \to 1$ we get by taking log, the rest are just applications of Stirling's formula).

(e) If $a, b < 0$ and $a, b \in Z$, then at some point the coefficients vanish, so the radius of convergence becomes ∞. Otherwise, we can easily see that for large n we have $\{x\}(1 - \{x\})(n - [|x|] - 1)! < |x(x+1)...(x+n-1)| < (n + [|x|])!$ (the right is obvious because every term on the RHS bounds its respective term on the LHS; for the left the idea is the same, the only difference is that now we need to take care of the negative terms and of the fractional parts). Actually all we need is that there exist A, B, C s.t. $A * (n - C)! < |x(x+1)...(x+n-1)| < B * (n+C)!$. From this immediately $(A * (n - C)!)^{1/n} < |x(x+1)...(x+n-1)|^{1/n} < (B * (n+C)!)^{1/n}$, so by the Sandwich theorem we have $|x(x+1)...(x+n-1)|^{1/n} = n/e * (1+o(1))$. Applying this for the coefficients in our series we get that $|a_n|^{1/n} = 1 + o(1)$, so the radius is 1.

(f) We have $|n! * (n+r)! * 2^{2*n+r}|^{1/2*n+r} = 2 * (1+o(1)) * n^{(n+1/2)/(2*n+r)} * (n+r)^{((n+r+1/2)/(2*n+r)} * e^{-1} \to \infty$ (because of the two terms which are $\Omega(\sqrt{n})$), so the radius of convergence is ∞.

Problem 7. *2.2*

We have $\int\limits_0^\infty \frac{\sin x}{x} dx = \frac{1}{2*i} \int\limits_0^\infty \frac{e^{i*x} - e^{-i*x}}{x} dx = \frac{1}{2*i} (\int\limits_0^\infty \frac{e^{i*x}-1}{x} dx - \int\limits_0^\infty \frac{e^{-i*x}-1}{x} dx = \frac{1}{2*i} \int\limits_{-\infty}^\infty \frac{e^{i*x}-1}{x} dx$. Now integrate along the big and small semicircles C_0 and C_1 shown below. For C_0: we have that $\int_{C_0} \frac{1}{x} dx = \pi * i$ and $|\int_{C_0} \frac{e^{i*x}}{x} dx| \leq 2 * |\int_{C_{00}} \frac{e^{i*x}}{x} dx| + |\int_{C_{01}} \frac{e^{i*x}}{x} dx|$ where C_{00} and C_{01} are shown below (C_{01} contains the part of C_0 that has points with imaginary parts more than a and C_{00} is one of the other 2 components). We have $|\int_{C_{00}} \frac{e^{i*x}}{x} dx| \leq \sup_{x \in C_{00}} (e^{i*x})/R * \int_{C_{00}} |dx| \leq e^{-a} * \pi$ and $|\int_{C_{01}} \frac{e^{i*x}}{x} dx| \leq |\int_{C_{01}} \frac{1}{x} dx| \leq \frac{1}{R} * C * a$ for some constant C (the constant C exists because the length of the curve approaches a as $a/R \to 0$). Thus, the integral of e^{i*x}/x over C_0 is bounded by $A * e^{-a} + B * a/R$ for some constants A and B. Pick R large and $a = \sqrt{R}$ and note that the above tends to 0. About the integral over C_1: We have $e^{i*x} - 1 = 1 + O(x)$ for $x \to 0$ (this is again from $\sin(x)/x \to 1$), so $|\int_{C_1} \frac{e^{i*x}-1}{x} dx| \leq O(1) * |\int_{C_1} dx| \to 0$ as $x \to 0$. Thus, we only care about the integral over C_{00} which is $-\pi * i$. Using Cauchy's theorem we get that our integral equals $\frac{1}{2*i}(-(\pi * i)) = \pi/2$.

Problem 8. *2.5*

Let $f = u(x,y) + iv(x,y)$ where u, v are real functions. Then we have $\displaystyle\int_T f(z)dz = \int_T (u_x + v_y)dx - i\int_T (v_x - u_y)dy$. Now by the Cauchy-Riemann equations and the Green theorem this integral is equal to 0.

Problem 9. *2.7*

By Cauchy's integral formula we have $2 * \pi * i * f'(0) = \int_{C_r} \frac{f(z)}{z^2} dz = -\int_{C_r} \frac{f(-z)}{z^2} dz$ (the second is just the transformation $z \to -z$). Thus, we have $|2*f'(0)| = |\frac{1}{2*\pi*i}\int_{C_r} \frac{f(z)-f(-z)}{z^2} dz| \leq \frac{1}{2*\pi} * \sup_{C_r} |\frac{f(z)-f(-z)}{z^2}| * \int_{C_r} |dz| \leq \frac{1}{2*\pi} * d/r^2 * 2 * \pi * r = d/r$ for any $r < 1$. Therefore $|2 * f'(0)| \leq d$ as the RHS tends to d as $r \to 1$.

Problem 10. *2.8*

Let $< 0 < d < 1/2$ be fixed. Let $D = k(x,d)$ be the disk with center x and radius d. From the Cauchy inequalities we get:
$|f^{(n)}(x)| \leq \frac{n!}{d^n} ||f(z)||_C$ where $C = \partial D$.
Now we use that $||f(z)||_C \leq A \sup\limits_{z \in C}(1+|z|)^n$ and this gives $|f^{(n)}| \leq A_n \sup\limits_{z \in C}(1+|z|)^n = A_n(1+|x|)^n \sup\limits_{z \in C}\left(\dfrac{\sup_{z\in C}(1+|z|)^n}{(1+|x|)^n}\right)$ where $A_n = \frac{An!}{d^n}$.
Now it remains to show that $\sup\limits_{z \in C} \dfrac{(1+|z|)^n}{(1+|x|)^n} \leq E_n$ for some constant E_n depending only on n. This follows from the trivial $\frac{1}{2} < \frac{1+|z|}{1+|x|} < 2$. The first inequality is equivalent to $2|z| + 1 > |x|$ which is true since $|z| > |x| - d$. The second inequality is equivalent to $2|x| + 1 > |z|$ which is true since $|z| < |x| + d$.

Problem 11. *2.13*

There are countably many coefficients c_n. Suppose that for every n it is true that every point z for which c_n of f at z is zero is isolated, i.e. that for every z there exists a circle around z s.t. c_n is non-zero for all points z' in this circle. We can choose the circles so that they do not intersect (take any collection of circles that satisfies the above rules and half all their radii). Then, since all of these circles contain a rational point, we can define an injection from the set of points for with $c_n = 0$ to Q, so this set is countable. But since there are countable many coefficients we get that the points for which any of the c_n is zero are countably many, which contradicts the condition that this is true for all points in C. Thus, there exists an n s.t. we have a sequence $z_1, z_2, ...$ which converges to z_0 s.t. $c_n = 0$ for all these points. Thus, $f^{(n)}(z_i) = 0 \forall i$. Thus, $f^{(n)}(z)$ is identically zero (Theorem 4.8). Thus, the power series for f is finite and f is a polynomial.

Problem 12. *Problem 1(a)*

Suppose f were regular at some point z on the unit circle. Clearly, for any open neighborhood of z there exists a $\theta = 2 * \pi * p/2^k$ s.t. $e^{i*\theta}$ is in this neighborhood (i.e. if the neighborhood contains a disk of radius ϵ, pick $k > \log \frac{4*\pi}{\epsilon}$). But this means that for $z = e^{i*\theta} * r$ and r close enough to 1 we must have $f(z) = g(z)$. But then $\lim f(z) = \lim g(z)$ for $r \to 1$, which is a contradiction because the left limit is infinite (because after some point all the coefficients become 1 and thus f will grow like $2^A/(1-r)$ for some A) and the right has to be finite (because g is analytic).

Problem 13. *Problem 2*

Expanding the right side we get $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty} z^n \sum_{k=0}^{\infty} z^{kn} = \sum_{n,k=1}^{\infty} z^{kn} = \sum_{n=1} d(n) z^n$. Where we use that $d(n)$ is the number of solutions of $ab = n$ in integers $a, b \geq 1$.

Now suppose $z = r$ for $0 < r < 1$. We have $f(r) = \displaystyle\sum_{n=1}^{\infty} \frac{r^n}{1-r^n} \geq c \int_1^{\infty} \frac{rx}{1-r^x} dx = \frac{1}{\ln r} \int_1^{\infty} \frac{dr^x}{1-r^x} = -\ln r \ln(1-r^x)|_1^{\infty} = \frac{\ln(1-r)}{\ln r}$. In a similar way we prove the inequality for all rational values of $\theta = 2\pi p/q$. Now since when $r \to \infty$ the right side of the inequality goes to infinity and the points $e^{i2\pi p/q}$ are dense on the unit circle we see that f can not be continued analitically beyond the unit disk.