Math 215A HW5

Solutions

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1 Problem 6.1

We have $1/\Gamma(s) = e^{\gamma s} s \prod_{n=1}^{\infty}(1+s/n)e^{-s/n}$ for all s. Thus, $\Gamma(s) = e^{-\gamma s}1/s \prod_{n=1}^{\infty} n/(n+s)e^{s/n}$ whenever $1/\Gamma(s) \neq 0 \Leftrightarrow s \neq 0, -1, -2, ...$ Now, note that $\gamma = \lim_{N\to\infty}(\sum_{n\leq N} 1/n - log(N))$, so $e^{-\gamma s} = \lim_{N\to\infty}(N^s e^{-s\sum_{n\leq N} 1/n})$. Thus, $\Gamma(s) = \lim_{N\to\infty}(N^s e^{-s\sum_{n\leq N} 1/n} 1/s \prod_{n=1}^{N} n/(n+s)e^{s/n}) = \lim_{N\to\infty} \frac{N^s N!}{s(s+1)...(s+N)}$, which is what we wanted.

2 Problem 6.3

Using $\Gamma(1/2) = \sqrt{\pi}$ and the previous exercise we get $\sqrt{\pi} = \lim_{N\to\infty} \frac{\sqrt{N}N!}{1/2(1/2+1)...(1/2+N)} = \lim_{N\to\infty} \frac{2^{N+1}\sqrt{N}N!}{(2N+1)!!} = \lim_{N\to\infty} \frac{2^{2N+1}\sqrt{N}(N!)^2}{(2N+1)!} = \lim_{N\to\infty} \frac{2^{2N}(N!)^2}{(2N+1)!} 2\sqrt{N} = \lim_{N\to\infty} \frac{2^{2N}(N!)^2}{(2N+1)!}\sqrt{2N+1}\sqrt{2}$,
which is what we wanted. For the other equality, see that
$\Gamma(s)\Gamma(s+1/2) = \lim_{N\to\infty} \frac{N^{2s+1/2}(N!)^2}{s(s+1/2)...(s+N)(s+N+1/2)} = \lim_{N\to\infty} \frac{2^{2N+2}N^{2s+1/2}(N!)^2}{s(s+1)..(s+2N+1)} =$
$\lim_{N\to\infty}(\frac{N^{2s}(2N+1)!}{2s(2s+1)..(2s+2N+1)} \frac{2^{2N+2}N^{2s+1/2}(N!)^2}{(2N+1)^{2s}(2N+1)!}) = \lim_{N\to\infty}(\frac{N^{2s}(2N+1)!}{2s(2s+1)..(2s+2N+1)} \frac{2^{2N}(N!)^2}{(2N+1)!}\sqrt{2N+1}\frac{4N^{2s+1/2}}{(2N+1)^{2s+1/2}}) =$
$\lim_{N\to\infty}(\frac{N^{2s}(2N+1)!}{2s(2s+1)..(2s+2N+1)} \frac{2^{2N}(N!)^2}{(2N+1)!}\sqrt{2N+1}2^{3/2-2s}(2N/(2N+1))^{2s+1/2} = \Gamma(2s)\sqrt{\pi}2^{1-2s}$,
where the last is just substituting the factors with their already known limits.

3 Problem 6.5

From the formula from 6.1, we can deduce that $|\Gamma(1/2+it)| = |\Gamma(1/2-it)|$ (because it holds for each of the corresponding pairs of terms from the limits). Thus, since $|\Gamma(1/2+it)\Gamma(1/2-it)| = |\pi/sin(\pi(1/2+it))|$, we have $|\Gamma(1/2+it)| = |\sqrt{\frac{\pi}{sin(\pi(1/2+it))}}| = |\sqrt{\frac{2\pi}{e^{i\pi(1/2+it)}-e^{-i\pi(1/2+it)}}}| = |\sqrt{\frac{2\pi}{e^{-\pi t}+e^{\pi t}}}|$.

4 Problem 6.7

(a) As the hint suggests, we have $\Gamma(\alpha)\Gamma(\beta) = \int_0^\infty \int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-t-s}dtds = \int_{(t,s)\in R_+^2} t^{\alpha-1}s^{\beta-1}e^{-t-s}$. Now looking at the function $(u,r) \to (ur, u(1-r))$ we can see that this is continuous, bijective and with a continuous inverse, which sends $R_+ \times (0,1)$ to R_+^2. The determinant of the derivative matrix is u, so we have $\int_{(t,s)\in R_+^2} t^{\alpha-1}s^{\beta-1}e^{-t-s} = \int_{(u,r)\in R_+\times(0,1)}(u(1-r))^{\alpha-1}(ur)^{\beta-1}e^{-u}u = \int_0^1 \int_0^\infty (u(1-r))^{\alpha-1}(ur)^{\beta-1}e^{-u}ududr = \int_0^1 (1-r)^{\alpha-1}r^{\beta-1} \int_0^\infty (u^{\alpha+\beta-1}e^{-u}dudr = \Gamma(\alpha+\beta)\int_0^1 (1-r)^{\alpha-1}r^{\beta-1}dr = \Gamma(\alpha+\beta)B(\alpha,\beta)$, which is what we wanted (we don't worry about dividing, since $Re(\alpha) > 0$ and $Re(\beta) > 0$).

(b) Do the transformation $x \to 1/x$ to see that $B(\alpha,\beta) = \int_0^1 (1-t)^{\alpha-1}t^{\beta-1}dt = \int_1^\infty (1-1/t)^{\alpha-1}1/t^{\beta-1}1/t^2dt = \int_1^\infty (t-1)^{\alpha-1}1/t^{\alpha+\beta}dt = \int_0^\infty t^{\alpha-1}1/(t+1)^{\alpha+\beta}dt.$

5 6.10

(a) Again, using the hint, we look at the function $f(w) = e^{-w}w^{z-1}$ along the contour from the picture. It is easy to see that the integral along the large quarter-circle is zero (we've already done that in one of the HW's, the idea is to split it into $Re(w) < a$ and $Re(w) > a$ for some proper a; in this case $a = R^{1-z-b}$ for some small b works). Clearly the integral along the small quarter-circle is also zero, since f is small as $w \to 0$. Thus, taking limits we have that $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt = \int_0^\infty e^{-it}(it)^{z-1}d(it) = i^z \int_0^\infty e^{-it}t^{z-1}dt$. We'll prove the result for $z \in (0,1)$, the general result will follow from analytic continuation. Since $e^{-it} = cos(t) - isin(t)$, we have shown $\Gamma(z)i^{-z} = \int_0^\infty cos(t)t^{z-1}dt - i\int_0^\infty sin(t)t^{z-1}dt$. The result follows from $i^{-z} = e^{-\pi/2z} = cos(\pi/2z) - isin(\pi/2z)$ and taking real and imaginary parts of the above identity.

(b) As is done in the textbook for the proof of the holomorphicity of Γ, we'll apply Theorem 5.4 of Chapter 2 to show that $F_\epsilon(z) = \int_\epsilon^{1/\epsilon} sin(t)t^{z-1}dt$ is holomorphic in the extended strip. Now, our integral is the limit of the above functions as $\epsilon \to 0$, so we just have to show uniform convergence. We'll do this by considering strips of the form $(-1+\delta, 1-\delta)$. Now $|\int_0^\epsilon sin(t)t^{z-1}dt| = |\int_0^\epsilon (sin(t)/t)t^z dt| \le C|\int_0^\epsilon t^z dt| = C|z||\epsilon^{z+1}| < C\epsilon^\delta$, where the first inequality is since $sin(x)/x \to 1$ as $x \to 0$; on the other hand $|\int_{1/\epsilon}^\infty sin(t)t^{z-1}dt| \le |\int_{1/\epsilon}^\infty sin(t)t^{-\delta}dt| \to 0$ since in (a) we've shown convergence of the integral for $z = 1-\delta$. Thus, we have uniform convergence, so $\int_0^\infty sin(t)t^{z-1}dt$ is holomorphic and agrees with $\Gamma(z)sin(\pi/2z)$ for $Re(z) \in (0,1)$, and thus agrees with it on the whole strip $Re(z) \in (-1+\delta, 1-\delta)$ for any small δ, which is what we wanted to show. The first identity is a consequence of $sin(x)/x \to 1$ and $\Gamma(x)x \to 1$ as $x \to 0$ and the second is just substituting with $z = -1/2$.

6 6.12

(a) Let n be a positive integer. We have $\Gamma(-n-1/2)G(n+3/2) = |1/\pi|$, so $|\Gamma(-n-1/2)| = \pi G(n+3/2) = \pi(n+1/2)G(n+1/2) = ... = \pi(1/2)(3/2)...(n+1/2)G(1/2) > \sqrt{\pi}/2n!$ (just compare term by term $1/2+k > k$). Thus, if $1/|\Gamma(s)|$ were $O(e^{c|s|})$, then $n!$ would be $O(e^{cn})$, which is not true (since it's $\Omega((n/e)^n)$ (e.g. integrate log and then exponentiate)).

(b) That's done in the chapter. If there were another such function, clearly it would have order 1, so we can apply Hadamard's formula to get the form of this function, which makes it clear that it differs from $1/\Gamma(z)$ just by a factor of e^{cz}, which implies that if this function were $O(e^{cn})$, then so would be $|1/\Gamma(z)|$, which is not true by (a).

7 6.15

For $x > 0$ we have $1/(e^x - 1) = \sum_{n=1}^{\infty} e^{-nx}$. Now, $1/\Gamma(s)\int_0^\infty x^{s-1}/(e^x-1)dx = 1/\Gamma(s)\int_0^\infty \sum_{n=1}^{\infty} e^{-nx}x^{s-1}dx = 1/\Gamma(s)\sum_{n=1}^{\infty}\int_0^\infty e^{-nx}x^{s-1}dx = 1/\Gamma(s)\sum_{n=1}^{\infty} n^{-s}\int_0^\infty e^{-nx}(xn)^{s-1}d(nx) = 1/\Gamma(s)\sum_{n=1}^{\infty} n^{-s}\Gamma(s) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s)$ (we used $Re(s) > 1$ for the convergence of $\sum 1/n^s$).

8 6.17

(a) Showing that the function is holomorphic for $Re(s) > 0$ can be done exactly the same way as it's shown in the chapter that Γ is holomorphic for $Re(s) > 0$ (all we need is to show uniform convergence of the integral, for which the only thing that is used is the faster than polynomial decay of f). For the othe part, suppose for some k it is true that $I(s) = (-1)^k/\Gamma(s+k)\int\limits_0^\infty f^{(k)}(x)x^{s+k-1}dx$ (true for $k = 0$). Now integrating by parts we get $I(s) = 1/\Gamma(s)\int\limits_0^\infty f^{(k)}(x)x^{s+k-1}dx = (-1)^k/\Gamma(s+k)(f^{(k)}(x)x^{s+k}/(s+k)|_0^\infty - 1/(s+k)\int\limits_0^\infty f^{(k+1)}(x)x^{s+k}dx = (-1)^{k+1}/\Gamma(s+k+1)\int\limits_0^\infty f^{(k+1)}(x)x^{s+k}dx$ (the other summand is zero, since all derivatives of f decay faster than any polynomial). Thus, by induction, the formula is true for any k. Now, since $\Gamma(s+k)$ is non-zero for $Re(s) > -k$ and the integral defines a holomorphic function for $Re(s) > -k$ (for the same reason for which $I(s)$ is holomorphic for $Re(s) > 0$, since f and $f^{(k)}$ satisfy the same decay property), so we have derived an analytic continuation for I over $Re(s) > -k$. This is true for any k, so we have an analytic continuation on the entire complex plane.

(b) Using the above formula for $k = n+1$ and plugging in $s = -n$ gives $I(-n) =$

$(-1)^{n+1} \int_0^\infty f^{(n+1)}(x)dx = (-1)^{n+1}(f^{(n)}(x)|_0^\infty) = (-1)^n f^{(n)}(0)$, where again the last is because of the fast decay of $f^{(n)}$.

9 Problem 2

By Corollary 2.6 we have $\zeta(s)-1/(s-1) = \sum_{n=1}^{\infty} \int_n^{n+1} (1/n^s - 1/x^s)dx$. Note that $\int_n^{n+1} (1/n^s - 1/x^s) = 1/n^s - \int_n^{n+1} (x/x^{s+1}) = 1/n^s - \int_n^{n+1} ((\{x\} + [x])/x^{s+1}) = 1/n^s - \int_n^{n+1} (n/x^{s+1} + \{x\}/x^{s+1}) = \int_n^{n+1} (1/n^s - n/x^{s+1}) - \int_n^{n+1} \{x\}/x^{s+1}$. Now, applying the mean value theorem for $f(x) = x^{-s-1}$ we get $|1/n^{s+1} - 1/x^{s+1}| \leq |s+1|/n^{Re(s)+2}$ and thus (multiplying by n and combining this with the similar result from Corollary 2.6) we get $|\int_n^{n+1} \{x\}/x^{s+1}| \leq c|s|/n^{Re(s)+1}$, which proves uniform convergence of the sum in the half-plane $Re(s) \geq \delta$. Thus, if we show that the identity holds for $Re(s) > 1$, then we are done by uniqueness of analytic continuation.

For $Re(s) > 1$ we just have $\int_n^{n+1} \{x\}/x^{s+1} = \int_n^{n+1} (x-[x])/x^{s+1} = \int_n^{n+1} 1/x^s - n \int_n^{n+1} 1/x^{s+1} = 1/(1-s)(1/(n+1)^{s-1} - 1/n^{s-1}) + 1/sn(1/(n+1)^s - 1/n^s)$. Summing over all n we get $\int_1^\infty \{x\}/x^{s+1} = 1/(s-1) - 1/s \sum_{n=1}^{\infty} n^{-s} = 1/(s-1) - 1/s\zeta(s)$, which is what we wanted. Thus, the identity holds for $Re(s) > 1$ and by the previous paragraph this implies that it holds for $Re(s) > 0$.

10 P3

We have (integrating by parts a bunch of times):

$\int_1^\infty (\frac{d^k}{x^k} Q_k(x) x^{-s-1}dx) = (\frac{d^{k-1}}{x^{k-1}} Q_k(x) x^{-s-1}dx)|_1^\infty + (s+1) \int_1^\infty (\frac{d^{k-1}}{x^{k-1}} Q_k(x) x^{-s-2}dx) = (\frac{d^{k-1}}{x^{k-1}} Q_k(x) x^{-s-1}dx)|_1^\infty + (s+1)(\frac{d^{k-2}}{x^{k-2}} Q_k(x) x^{-s-2}dx)|_1^\infty + (s+1)(s+2) \int_1^\infty (\frac{d^{k-2}}{x^{k-2}} Q_k(x) x^{-s-3}dx) = \ldots = \sum_{i=1}^{k-1} (s+i-1)!/s!(\frac{d^{k-i}}{x^{k-i}} Q_k(x) x^{-s-i}dx)|_1^\infty + (s+k)!/s! \int_1^\infty Q(x) x^{-s-k-1}dx$. Each of the summands is some polynomial in s, the integral at the end converges since $Re(s+k+1) > 1$. Thus, the right-hand side is a holomorphic function for $Re(s) > -k$ (it is holomorphic since it is a sum of polynomials and $(s+k)!/s! \int_1^\infty Q(x)x^{-s-k-1}dx$, which is holomorphic by the last problem), thus, it is an analytic continuation of the function on the left-hand side. Substituting the expression from the problem with our derived expression gives an analytic continuation of $\zeta(s)$ for $Re(s) > -k$.