MATH 171: WRITING IN MAJOR ASSIGNMENT

May 6, 2010

In this writing assignment we will explore the p-adic topology on rational numbers. Here p will denote a prime number. If you need to, you may assume that all natural numbers have a unique factorization into primes. In class we discussed special cases of this topology taking $p = 7$ several times. Below I will give several questions for you to ponder. Your write-up should include solutions to these, but they are really just intended to get you thinking. You should feel free to expand on these problems, and also to make a nice synthesis. Your final draft should read like a chapter in a book trying to explain this p-adic topology rather than a homework assignment solving a bunch of problems.

Let us recall first the p-adic absolute value which was defined in class. Given any integer $n \neq 0$ we write (uniquely) $n = p^a b$ where b is an integer with p not dividing b. Then $|n|_p$ is defined to be p^{-a}. We define $|0|_p = 0$ and thus we have defined the p-adic absolute value on all integers. To extend this definition to all rational numbers, write $x \in \mathbb{Q}$ as $x = r/s$ where r is an integer, and s is a natural number and with r and s in reduced form (that is, r and s have no common factors). Then $|x|_p$ is defined to be $|r|_p/|s|_p$.

1. Prove that $\mathbb{Q}$ forms a metric space with the distance function d defined by $d(x, y) = |x - y|_p$.

2. Discuss, giving examples, open and closed subsets of $\mathbb{Q}$ with the p-adic metric. Is the set of integers open or closed or both or neither? Prove that if O_1 and O_2 are two open neighborhoods then either they are disjoint or one is contained in the other.

3. Discuss, giving examples, sequences of points in $\mathbb{Q}$ and whether they converge in the p-adic metric. One specific example: given $z \in \mathbb{Q}$ with $|z|_p < 1$ prove that the sequence $1, 1 + z, 1 + z + z^2, 1 + z + z^2 + z^3, \ldots$ is convergent and find its limit.

4. Consider the case $p = 7$. Show that there exist integers $a_n \in [0, 6]$ for each $n = 0, 1, 2, \ldots$ with

$$\Big|2 - \Big(\sum_{j=0}^{n} a_j 7^j\Big)^2\Big|_7 \leq 7^{-n}.$$

The sequence $x_n = \sum_{j=0}^{n} a_j 7^j$ is a Cauchy-sequence in the 7-adics. Explain why there is no rational number x which is the limit (in the 7-adics) of this sequence. What is the limit of x_n^2? What do you think should be the limit of x_n (in some other space)?

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5. Discuss the completion of the metric space $\mathbb{Q}$ with the p-adic metric. This completion is denoted by $\mathbb{Q}_p$. In analogy with decimal expansions explain what the elements of $\mathbb{Q}_p$ look like. Prove that $\sqrt{2}$ exists in $\mathbb{Q}_7$. Does $\sqrt{2}$ exist in $\mathbb{Q}_5$? Prove that $\sqrt{-1}$ exists in $\mathbb{Q}_5$.

6. (Extra credit). Any other topics on the p-adic that are related to the general theory of metric spaces. Some ideas:

(i) Let $\mathbb{Z}_p$ denote the elements of $\mathbb{Q}_p$ with p-adic absolute value ≤ 1. Prove that $\mathbb{Z}_p$ is compact.

(ii) Is $\mathbb{Q}_p$ connected?

(iii) Prove that a series $\sum_{n=1}^{\infty} a_n$ with $a_n \in \mathbb{Q}_p$ converges if and only if $|a_n|_p \to 0$ as $n \to \infty$.