Homework 8 Solutions

Math 171, Spring 2010
Henry Adams

44.2. (a) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.
(b) Prove that $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$.

Solution. (a) Given $\epsilon > 0$, pick $\delta = \epsilon^2$. First note that $|\sqrt{x} - \sqrt{y}| \leq |\sqrt{x} + \sqrt{y}|$. Hence if $|x - y| < \delta = \epsilon^2$, then we have

$$|\sqrt{x} - \sqrt{y}|^2 \leq |\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| = |x - y| < \epsilon^2,$$

hence $|\sqrt{x} - \sqrt{y}| < \epsilon$. This shows that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

(b) Pick $\epsilon = 1$. Given any $\delta > 0$, pick $x > 0$ such that $\frac{3\delta x^2}{2} > 1$. Then $d(x + \frac{\delta}{2}, x) < \delta$ but we have

$$d(f(x + \frac{\delta}{2}), f(x)) = |(x + \frac{\delta}{2})^3 - x^3| = |\frac{3\delta x^2}{2} + \frac{3\delta^2 x}{2^2} + \frac{\delta^3}{2^3}| \geq \frac{3\delta x^2}{2} > 1.$$

This shows that $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$.

44.5. Let M_1, M_2, and M_3 be metric spaces. Let g be a uniformly continuous function from M_1 into M_2, and let f be a uniformly continuous function from M_2 into M_3. Prove that $f \circ g$ is uniformly continuous on M_1.

Solution. Let $\epsilon > 0$. Since f is uniformly continuous, there exists some $\delta > 0$ such that $d_2(x, y) < \delta$ implies $d_3(f(x), f(y)) < \epsilon$ for all $x, y \in M_2$. Since g is uniformly continuous, there exists some $\gamma > 0$ such that $d_1(z, w) < \gamma$ implies $d_2(g(z), g(w)) < \delta$ for all $z, w \in M_1$. Putting these together, we get that $d_1(z, w) < \gamma$ implies $d_3(f(g(z)), f(g(w)) < \epsilon$ for all $z, w \in M_1$. Hence $f \circ g$ is uniformly continuous on M_1.

44.7. A contraction mapping on M is a function f from the metric space (M, d) into itself satisfying $d(f(x), f(y)) \leq cd(x, y)$ for some c, $0 \leq c < 1$ and all x and y in M.

(a) Prove that a contraction mapping on M is uniformly continuous on M.

Solution. Let f be a contraction mapping on M. Given $\epsilon > 0$, pick $\delta = \epsilon$. Then if $d(x, y) < \delta$, we have $d(f(x), f(y)) \leq cd(x, y) < c\delta = c\epsilon < \epsilon$. Hence f is uniformly continuous on M.

(b) Give an example of a contraction mapping from $\mathbb{R}$ onto $\mathbb{R}$.

Solution. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x}{2}$. Note that f maps onto $\mathbb{R}$ because $f(2x) = x$ for any $x \in \mathbb{R}$. Note that

$$d(f(x), f(y)) = |\frac{x}{2} - \frac{y}{2}| = \frac{1}{2}|x - y| = \frac{1}{2}d(x, y)$$

so f is a contraction mapping with $c = \frac{1}{2}$.

(c) Prove that there is no contraction mapping from a compact metric space (with more than one point) onto itself.

Solution. Suppose for a contradiction that $f : M \to M$ is a contraction mapping from compact space M onto itself, where M is not a point. Let $x \neq y \in M$ be arbitrary. Because f is onto, we have $x = f(x')$ and $y = f(y')$ for some $x' \neq y' \in M$. Note

$$d(x, y) = d(f(x'), f(y') \leq cd(x', y') < d(x', y') \leq \operatorname{diam} M.$$

Hence $d(x, y) < \operatorname{diam} M$ for all $x, y \in M$. Since M is compact, this contradicts Exercise 43.4. Hence there is no contraction mapping from a compact metric space (with more than one point) onto itself.

45.1. Prove that $l^1,\ l^2,\ c_0,\ l^\infty$, and H^∞ are connected metric spaces.

Solution. In each case, follow the proof of Theorem 45.7. Given $x, y \in l^1,\ l^2,\ c_0,\ l^\infty$, or H^∞, define $f(t)$ for $0 \leq t \leq 1$ as in Theorem 45.7. The only change in the proof is showing that f is a continuous function from $[0, 1]$ into $l^1,\ l^2,\ c_0,\ l^\infty$, or H^∞. For the case of l^1, use Theorem 23.1. For the case of l^2, use Theorem 23.1 and Theorem 36.6.

45.2. (a) Give an example of a subset of $\mathbb{R}$ which is connected but not compact.

Solution. $(0, 1)$ is connected by Corollary 45.4 and is not compact because it is not closed.

(b) Give an example of a subset of $\mathbb{R}$ which is compact but not connected.

Solution. $[0, 1] \cup [2, 3]$ is not connected by Corollary 45.4 and is compact by Theorem 43.9.

(c) Characterize the compact, connected subsets of $\mathbb{R}$.

Solution. By Corollary 45.4, a subset of $\mathbb{R}$ is connected if and only if X is empty, a point, or an interval. By Theorem 43.9, a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded. Therefore, a subset of $\mathbb{R}$ is compact and connected if and only if it is of the form $\emptyset$, $\{a\}$ for some $a \in \mathbb{R}$, or $[a, b]$ for some $a, b \in \mathbb{R}$.

45.4. Let M be a metric space. Prove that the following are equivalent.
(a) M is not connected.
(b) There exist nonempty subsets X and Y of M such that $M = X \cup Y$, $\overline{X} \cap Y = \emptyset = X \cap \overline{Y}$.

Solution. To see (a) implies (b), let X and Y be the closed sets C and D from Theorem 45.2(iii). Note that the closure of a closed set is itself.

To see (b) implies (a), note that since $M = X \cup Y$ and $X \cap Y \subset \overline{X} \cap Y = \emptyset$ we have $Y = X^c$. Hence $\overline{X} \cap X^c = \emptyset = X \cap \overline{X^c}$. The first equation tells us that X contains all its limit points, hence is closed. The second tells us that X^c contains all its limit points, hence is closed. So letting C and D in Theorem 45.2(iii) be X and Y, we see that M is not connected.

46.1. Prove that every finite subset of a metric space is complete.

Solution. Let X be a finite subset of a metric space. Let $c = \min\{d(x, y) : x \neq y \in X\}$. Note that $c > 0$. Suppose that $\{x_n\}$ is a Cauchy sequence in X. Then there exists a positive integer N such that if $m, n \geq N$, then $d(x_m, x_n) < c$. Therefore, if $n \geq N$ then $x_n = x_N$, and so $\{x_n\}$ converges to x_N. This shows that X is complete.

46.4. Give an example of a complete metric space which is not connected.

Solution. $[0, 1] \cup [2, 3]$ is not connected by Corollary 45.4. To see that $[0, 1] \cup [2, 3]$ is complete, let $\{x_n\}$ be a Cauchy sequence in $[0, 1] \cup [2, 3]$. Since $\mathbb{R}$ is complete, $\{x_n\}$ converges to some $x \in \mathbb{R}$. Since $[0, 1] \cup [2, 3]$ is closed, it contains all its limit points, and so $x \in [0, 1] \cup [2, 3]$. This shows $\{x_n\}$

converges to some $x \in [0,1] \cup [2,3]$, and so $[0,1] \cup [2,3]$ is complete.

46.7. Prove that a compact metric space is complete.

Solution. Let M be a compact metric space and let $\{x_n\}$ be a Cauchy sequence in M. By Theorem 43.5, there exists a convergent subsequence $\{x_{n_k}\}$. Let $x = \lim_{k\to\infty} x_{n_k}$. Since $\{x_n\}$ is Cauchy, there exists some N such that $m, n \geq N$ implies $d(x_m, x_n) < \frac{\epsilon}{2}$. Since $x = \lim_{k\to\infty} x_{n_k}$, there exists some K with $n_K > N$ such that $k \geq K$ implies $d(x, x_{n_k}) < \frac{\epsilon}{2}$. Then for $n \geq n_K$, we have

$$d(x, x_n) \leq d(x, x_{n_K}) + d(x_{n_K}, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\{x_n\}$ converges to x, showing that M is complete.

46.8. Prove that l^2, c_0, and l^∞ are complete metric spaces.

Solution. In all three cases, one can adapt the proof of Theorem 46.5, most of it word-for-word! We give the details for the case l^2. The arguments for c_0 and l^∞ are similar.

Let $\{a^{(n)}\}$ be a Cauchy sequence of points in l^2. Let $\epsilon > 0$. There exists a positive integer N such that if $m, n \geq N$, then

$$(1) \qquad |a_i^{(m)} - a_i^{(n)}| \leq \sqrt{\sum_{k=1}^{\infty} (a_k^{(m)} - a_k^{(n)})^2} = d(\{a^{(m)}\}, \{a^{(n)}\}) < \epsilon$$

for any positive integer i. Thus for any positive integer i, $\{a_i^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. By Theorem 19.3, $\{a_i^{(n)}\}_{n=1}^{\infty}$ is convergent. We let $a_i = \lim_{n\to\infty} a_i^{(n)}$.

From equation (1), we have

$$\sum_{k=1}^{\infty} (a_k^{(n)})^2 = d(a^{(n)}, 0) \leq d(a^{(n)}, a^{(N)}) + d(a^{(N)}, 0) = \sum_{k=1}^{\infty} (a_k^{(n)} - a_k^{(N)})^2 + \sum_{k=1}^{\infty} (a_k^{(N)})^2 < \epsilon^2 + \sum_{k=1}^{\infty} (a_k^{(N)})^2$$

if $n \geq N$. Thus for any positive integer p, if $n \geq N$,

$$\sum_{k=1}^{p} (a_k^{(n)})^2 < T$$

where $T = \epsilon^2 + \sum_{k=1}^{\infty} (a_k^{(N)})^2$. Taking the limit as $n \to \infty$, we have

$$\sum_{k=1}^{p} (a_k)^2 \leq T$$

for every positive integer p. By Theorem 24.1, $\{a_k\} \in l^2$.

Again using equation (1), we have for any positive integer p,

$$\sqrt{\sum_{k=1}^{p} (a_k^{(m)} - a_k^{(n)})^2} < \epsilon$$

if $m, n \geq N$. Taking the limit as $m \to \infty$, we have

$$\sqrt{\sum_{k=1}^{p} (a_k - a_k^{(n)})^2} \leq \epsilon$$

for $n \geq N$. Taking the limit as $p \to \infty$, we have

$$d(\{a_k\}, \{a_k^{(n)}\}) = \sqrt{\sum_{k=1}^{\infty} (a_k - a_k^{(n)})^2} \leq \epsilon$$

if $n \geq N$, and hence $\{a^{(n)}\}_{n=1}^{\infty}$ converges to $\{a_k\}$ in l^2.

60.2. Let $f_n(x) = 1/(1+n^2x^2)$ and $g_n(x) = nx(1-x)^n$, $x \in [0,1]$. Prove that $\{f_n\}$ and $\{g_n\}$ converge pointwise but not uniformly on $[0,1]$.

Solution. Show that $\{f_n\}$ converges pointwise to f on $[0,1]$, where

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \leq 1. \end{cases}$$

Since f is not continuous, $\{f_n\}$ cannot converge uniformly because otherwise this would contradict Theorem 60.4.

The sequence $\{g_n\}$ converges pointwise to the zero function. If $x = 0$ or $x = 1$ then this is clear, as we have $g_n(x) = 0$ for all n and so $\lim_{n\to\infty} g_n(x) = 0$. So fix $x \in (0,1)$. Note that

$$\lim_{n\to\infty} \left| \frac{g_{n+1}(x)}{g_n(x)} \right| = \lim_{n\to\infty} \frac{(n+1)x(1-x)^{n+1}}{nx(1-x)^n} = \lim_{n\to\infty} \frac{n+1}{n}(1-x) = 1-x < 1.$$

Hence the series $\sum_{n=1}^{\infty} g_n(x)$ converges absolutely by Theorem 26.6 (Ratio Test), and so $\lim_{n\to\infty} g_n(x) = 0$ by Theorem 22.3. So $\{g_n\}$ converges pointwise to the zero function.

By the comments after Definition 603, if $\{g_n\}$ were to converge uniformly on $[0,1]$, then it would need to converge to the zero function. However, note that $g_n(\frac{1}{n}) = (1-\frac{1}{n})^n$. Therefore $\lim_{n\to\infty} g_n(\frac{1}{n}) = \lim_{n\to\infty}(1-\frac{1}{n})^n = \frac{1}{e}$. Pick $\epsilon = \frac{1}{2e}$. So there cannot exist a positive integer N such that if $n \geq N$ then $|g_n(x) - 0| < \frac{1}{2e} = \epsilon$ for all $x \in X$. This shows that $\{g_n\}$ does not converge uniformly to the zero function, and hence does not converge uniformly on $[0,1]$.

60.5. Let $\{f_n\}$ be a sequence of bounded functions on a set X. Prove that if $\{f_n\}$ converges uniformly to f on X, then f is bounded. Give an example to show that this statement is false if uniform convergence is replaced by pointwise convergence.

Solution. Pick $\epsilon = 1$. Since $\{f_n\}$ converges uniformly to f, by Definition 60.3 there exists a positive integer N such that $|f_N(x) - f(x)| < 1$ for all $x \in X$. Since f_N is bounded, there exists some M such that $|f_N(x)| < M$ for all $x \in X$. By the triangle inequality, we have $|f(x)| < M+1$ for all $x \in X$. Therefore f is bounded.

For the example, let $X = (0,1]$ and let

$$f_n(x) = \begin{cases} n & 0 < x \leq \frac{1}{n} \\ \frac{1}{x} & \frac{1}{n} < x \leq 1. \end{cases}$$

Then each f_n is bounded, but $\{f_n\}$ converges pointwise to the function $\frac{1}{x}$, which is unbounded on $(0,1]$.