Homework 7 Solutions

Math 171, Spring 2010
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42.1. Prove that none of the spaces $\mathbb{R}^n$, l^1, l^2, c_0, or l^∞ is compact.

Solution. Let $X = \mathbb{R}^n$, l^1, l^2, c_0, or l^∞. Let $0 = (0, \dots, 0)$ in the case $X = \mathbb{R}^n$ and let $0 = (0, 0, \dots)$ in the case $X = l^1$, l^2, c_0, or l^∞. For $n \in \mathbb{P}$, let $B_n(0)$ be the ball of radius n about 0 with respect to the relevant metric on X. Note that $\mathcal{U} = \{B_n(0) : n \in \mathbb{P}\}$ is an open cover of X. However, if $\mathcal{U}$ had a finite subcover, then we would have $X = B_N(0)$ for some $N \in \mathbb{P}$. This is a contradiction in all cases. In the case of $X = \mathbb{R}^N$, note that $(N+1, 0, \dots, 0) \notin B_N(0)$. In the case $X = l^1$, l^2, c_0, or l^∞, note that $(N+1, 0, \dots) \notin B_N(0)$. Hence none of the spaces $\mathbb{R}^n$, l^1, l^2, c_0, or l^∞ is compact.

42.3. Let $X_1, \dots, X_n$ be a finite collection of compact subsets of a metric space M. Prove that $X_1 \cup X_2 \cup \dots \cup X_n$ is a compact metric space. Show (by example) that this result does not generalize to infinite unions.

Solution. Let $\mathcal{U}$ be an open cover of $X_1 \cup X_2 \cup \dots \cup X_n$. Then $\mathcal{U}$ is an open cover of X_i for each $1 \leq i \leq n$. Since each X_i is compact, there is a finite subcover U_i^* of X_i for each i. Let $\mathcal{U}^* = \cup_{i=1}^n \mathcal{U}_i^*$. Then $\mathcal{U}^*$ is finite collection as it is a finite union of finite collections. Also, $\mathcal{U}^*$ covers X_i for all i as $\mathcal{U}_i^*$ covers X_i for each i. Therefore $\mathcal{U}^*$ is a finite subcollection of $\mathcal{U}$ covering $X_1 \cup X_2 \cup \dots \cup X_n$, and so $X_1 \cup X_2 \cup \dots \cup X_n$ is compact.

To see that this result does not generalize to infinite unions, let $M = \mathbb{R}$ and let $X_n = [n-1, n]$ for all $n \in \mathbb{P}$. Then each X_n is compact, but $\cup_{n=1}^\infty X_n = \cup_{n=1}^\infty [n-1, n] = [0, \infty)$ is not compact.

42.5. A collection $\mathcal{C}$ of subsets of a set X is said to have the finite intersection property if whenever $\{C_1, \dots, C_n\}$ is a finite subcollection of $\mathcal{C}$, we have $C_1 \cap C_2 \cap \dots \cap C_n \neq \emptyset$. Prove that a metric space M is compact if and only if whenever $\mathcal{C}$ is a collection of closed subsets of M having the finite intersection property, we have $\cap \mathcal{C} \neq \emptyset$.

Solution. First, suppose that M is compact. Let $\mathcal{C}$ be a collection of closed subsets of M having the finite intersection property. Let $\mathcal{U} = \{C^c : C \in \mathcal{C}\}$. Then $\mathcal{U}$ is a collection of open sets. Suppose for a contradiction that $\cup \mathcal{U} = M$. Then since M is compact, there exists some finite subcover $\mathcal{U}^*$ of $\mathcal{U}$. Label the sets in $\mathcal{U}^*$ as $\mathcal{U}^* = \{C_1^c, \dots, C_n^c\}$ with $C_i \in \mathcal{C}$ for all i. Since $\mathcal{C}$ has the finite intersection property, we have $C_1 \cap C_2 \cap \dots \cap C_n \neq \emptyset$. Taking complements, we get $C_1^c \cup C_2^c \cup \dots \cup C_n \neq M$, contradicting the fact that $\mathcal{U}^*$ is a cover of M. Hence it must be that $\cup \mathcal{U} \neq M$, and taking complements gives $\cap \mathcal{C} \neq \emptyset$.

Next, suppose whenever $\mathcal{C}$ is a collection of closed subsets of M having the finite intersection property, we have $\cap \mathcal{C} \neq \emptyset$. Let $\mathcal{U}$ be an open cover of M. Let $\mathcal{C} = \{U^c : U \in \mathcal{U}\}$, so $\mathcal{C}$ is a collection of closed subsets. Since $\mathcal{U}$ is an open cover, we have $\cup \mathcal{U} = M$ hence $\cap \mathcal{C} = \emptyset$. By assumption, this means that $U_1^c \cap \dots \cap U_n^c = \emptyset$ for some finite subset of $\mathcal{C}$. Taking complements, we get that $U_1 \cup \dots \cup U_n = M$ for some finite subset of $\mathcal{U}$. This shows that M is compact.

42.10. Let $\{X_n\}$ be a sequence of compact subsets of a metric space M with $X_1 \supset X_2 \supset X_3 \supset \cdots$. Prove that if U is an open set containing $\cap X_n$, then there exists $X_n \subset U$.

Solution. Note $M = U \cup (\cup_{n=1}^\infty X_n^c)$ so $X_1 \subset (U) \cup (\cup_{n=2}^\infty X_n^c)$. Hence $U, X_1^c, X_2^c, X_3^c, X_4^c \dots$ is an open cover of the compact space X_1. By definition of compactness, there exists some finite subcover. Since $X_2^c \subset X_3^c \subset X_4^c \subset \dots$, this means there exists some n such that $X_1 \subset U \cup X_n^c$. Hence

$$X_n = X_1 \cap X_n \subset (U \cup X_n^c) \cap X_n = U \cap X_n \subset U.$$

42.12. A contractive mapping on M is a function f from the metric space (M, d) into itself satisfying $d(f(x), f(y)) < d(x, y)$ whenever $x, y \in M$ with $x \neq y$. Prove that if f is a contractive mapping on a compact metric space M, there exists a unique point $x \in M$ with $f(x) = x$.

Solution. Suppose there does not exist such a fixed point x with $f(x) = x$. Then the function $g(x) = d(f(x), x)$ positive. To see that g is continuous, use the fact that f is continuous (which follows since f satiesfies the contractive mapping property) and the triangle inequality. Since M is compact, by Corollary 42.7 there exists $c \in M$ such that $g(c) \leq g(x)$ for all $x \in M$. As g is positive, this means that $g(c) > 0$. However, note

$$g(f(c)) = d(f(f(c)), f(c)) < d(f(c), c) = g(c).$$

This is a contradiction. Therefore, there must exist a fixed point $x \in M$ with $f(x) = x$. This fixed point must be unique, for if there were some $y \neq x$ with $f(y) = y$, then we would have $d(f(x), f(y)) = d(x, y)$, which contradicts our hypotheses.

43.1. Prove that the set $\{x \in M : d(x, 0) = 1\}$ is closed and bounded in M, but not compact if M is l^2, c_0, or l^∞.

Solution. Let $f(x) = d(x, 0)$, which is a continuous function by Theorem 40.3. So $\{x \in M : d(x, 0) = 1\} = f^{-1}(\{1\})$ is the continuous preimage of a closed set, hence closed by Theorem 40.5(ii).

Note that $d(y, z) \leq 2$ for all $y, z \in \{x \in M : d(x, 0) = 1\}$, as $d(y, z) \leq d(y, 0) + d(0, z) = 1 + 1 = 2$. Hence $\{x \in M : d(x, 0) = 1\}$ is bounded by Definition 43.6.

Let $\delta^{(k)}$ in l^2, c_0, or l^∞ be given by

$$\delta_n^{(k)} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases}$$

Check that $\{\delta^{(k)}\}_{k=1}^\infty$ is a sequence of points in l^2, c_0, or l^∞ that has no convergent subsequence. Therefore l^2, c_0, and l^∞ are not compact by Theorem 43.5.

43.4. If (M, d) is a bounded metric space, we let $\operatorname{diam} M = \operatorname{lub}\{d(x, y) : x, y \in M\}$. Prove that if (M, d) is a compact metric space, there exist $x, y \in M$ such that $d(x, y) = \operatorname{diam} M$.

Solution. I will give two solutions.

First solution: For each $x \in M$, define $f(x) = \max\{d(x, y) : y \in M\}$, where this maximum is realized by Theorem 40.3 and Corollary 42.7. Let $y_x \in M$ be a point such that $f(x) = d(x, y_x)$.

We want to show that f is continuous. Let $\epsilon > 0$. Suppose $d(x, x') < \epsilon$. It must be that $d(x, y_x) < \epsilon + d(x', y_{x'})$, for otherwise we would have

$$d(x, y_{x'}) \leq d(x, x') + d(x', y_{x'}) < \epsilon + d(x', y_{x'}) \leq d(x, y_x),$$

contradicting the choice of y_x. Similarly, it must be that $d(x', y_{x'}) < \epsilon + d(x, y_x)$. Together, these two inequalities show that $|d(x, y_x) - d(x', y_{x'})| < \epsilon$. Hence

$$|f(x) - f(x')| = |d(x, y_x) - d(x', y_{x'})| < \epsilon.$$

This shows f is continuous.

Therefore, we apply Corollary 42.7 to see that there exists some $c \in M$ such that $f(c) \geq f(x)$ for all $x \in M$. Hence

$$d(c, y_c) = f(c) \geq f(x) = \max\{d(x, y) : y \in M\}$$

for all $x \in M$. This shows that $d(c, y_c) = \text{lub}\{d(x,y) : x, y \in M\} = \text{diam}\, M$.

Second solution: Consider the product metric space $(M \times M, d')$, where d' is defined by $d'[(x_1, x_2), (y_1, y_2)] = d(x_1, x_2) + d(y_1, y_2)$ as in Exercise 35.8. Since (M, d) is compact, by Exercise 43.2 it follows that $(M \times M, d')$ is compact. Show that $d : M \times M \to \mathbb{R}$ is continuous, using the definition of d' and the triangle inequality. So Corollary 42.7 tells us that there exist points $(c, d) \in M \times M$ such that $d(c, d) \geq d(x, y)$ for all x, y in M. Hence $d(c, d) = \text{diam}\, M$.

43.7. Let X be a compact subset of a metric space M. If $y \in X^c$, prove that there exists a point $a \in X$ such that $d(a, y) \leq d(x, y)$ for all $x \in X$. Give an example to show that the conclusion may fail if "compact" is replaced by "closed."

Solution. Let $f(x) = d(x, y)$. By Theorem 40.3, the function f is continuous. By Corollary 42.7, there exists a point $a \in X$ such that $d(a, y) \leq d(x, y)$ for all $x \in X$.

To see that the conclusion may fail if "compact" is replaced by "closed," let $M = l^\infty$. Let $\delta^{(k)} \in l^\infty$ be given by

$$\delta_n^{(k)} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k, \end{cases}$$

and let $X = \{\delta^{(k)} : k \in \mathbb{P}\}$. Note that X contains its limit points and is therefore closed. Let $y = (-1, -\frac{1}{2}, \ldots, -\frac{1}{n}, \ldots)$. So $d(\delta^{(k)}, y) = 1 + \frac{1}{k}$ for all $k \in \mathbb{P}$, which implies that there does not exist a fixed $k_0 \in \mathbb{P}$ such that $d(\delta^{(k_0)}, y) \leq d(\delta^{(k)}, y)$ for all $k \in \mathbb{P}$.

44.1. Give an example of metric spaces M_1 and M_2 and a continuous function f from M_1 onto M_2 such that M_2 is compact, but M_1 is not compact.

Solution. Let $M_1 = \mathbb{R}$, let M_2 be the trivial metric space $\{0\}$ consisting of a single point, and let $f : \mathbb{R} \to \{0\}$ be given by $f(x) = 0$ for all $x \in \mathbb{R}$. Check that f is a continuous function. Note that $M_2 = \{0\}$ is compact, but $M_1 = \mathbb{R}$ is not compact.

44.6(a,b,c). Let f be a one-to-one function from a metric space M_1 onto a metric space M_2. If f and f^{-1} are continuous, we say that f is a homeomorphism and that M_1 and M_2 are homeomorphic metric spaces.

(a) Prove that any two closed intervals of $\mathbb{R}$ are homeomorphic.

Solution. Let $[a, b]$ and $[c, d]$ be any two closed intervals of $\mathbb{R}$. Define $f : [a, b] \to [c, d]$ by $f(x) = \frac{d-c}{b-a}(x - a) + c$. Check that f is one-to-one and onto, and that $f^{-1}[c, d] \to [a, b]$ is given by $f^{-1}(x) = \frac{b-a}{d-c}(x - c) + a$. Check that f and f^{-1} are continuous functions, and hence $[a, b]$ and $[c, d]$ are homeomorphic.

(b) Prove (a) with "closed" replaced by "open"; with "closed" replaced by "half-open".

Solution. When "closed" is replaced by "open", the argument given in (a) works after replacing $[a, b]$ and $[c, d]$ with (a, b) and (c, d), respectively.

When "closed" is replaced by "half-open," there are four cases. If the two intervals are $(a, b]$ and $(c, d]$ or $[a, b)$ and $[c, d)$, then the argument given in (a) carries over. If the two intervals are $(a, b]$ and $[c, d)$ or $[a, b)$ and $(c, d]$, then define f by $f(x) = -\frac{d-c}{b-a}(x - a) + d$, and proceed as above.

(c) Prove that a closed interval is not homeomorphic to either an open interval or a half-open interval.

Solution. I will prove the following claim: if two spaces M_1 and M_2 are homeomorphic, then M_1 is compact if and only if M_2 is compact. For the proof, note that if M_1 is compact, then $M_2 = f(M_1)$

is compact by Theorem 44.1. Conversely, if M_2 is compact, then $M_1 = f^{-1}(M_2)$ is compact by Theorem 44.1.

Since a closed interval is compact but an open interval or a half-open interval is not compact, our claim shows that a closed interval is not homeomorphic to either an open interval or a half-open interval.

44.8. Let X be a compact subset of $\mathbb{R}$, and let f be a real-valued function on X. Prove that f is continuous if and only if $\{(x, f(x)) : x \in X\}$ is a compact subset of $\mathbb{R}^2$.

Solution. First, suppose f is continuous. Let $\{(x_n, f(x_n))\}$ be any sequence of points in $\{(x, f(x)) : x \in X\}$. Then $\{x_n\}$ is a sequence of points in X. Since X is compact, by Theorem 43.5 there is a convergent subsequence $\{x_{n_k}\}$ with $\lim_{k\to\infty} x_{n_k} = x \in X$. Since f is continuous, by Theorem 40.2 we have $\lim_{k\to\infty} f(x_{n_k}) = f(x)$. Hence $\lim_{k\to\infty}(x_{n_k}, f(x_{n_k})) = (x, f(x))$ by Theorem 37.2. Hence $\{(x, f(x)) : x \in X\}$ is compact by Theorem 43.5.

Conversely, suppose that $\{(x, f(x)) : x \in X\}$ is a compact subset of $\mathbb{R}^2$. Let $\{x_n\}$ be any sequence of points in X with $\lim_{n\to\infty} x_n = x \in X$. Consider the sequence of points $\{(x_n, f(x_n))\}$ in $\{(x, f(x)) : x \in X\}$. Let U be any open neighborhood about $(x, f(x))$ in $\mathbb{R}^2$. Suppose for a contradiction that $\{(x_n, f(x_n))\}$ is not eventually inside U. Then there is a subsequence $\{(x_{n_k}, f(x_{n_k}))\}$ which lies in $\{(x, f(x)) : x \in X\} \setminus U$. Note $\{(x, f(x)) : x \in X\} \setminus U$ is a closed subset of the compact set $\{(x, f(x)) : x \in X\}$ and is hence compact. So $\{(x_{n_k}, f(x_{n_k}))\}$ has a convergent subsequence in $\{(x, f(x)) : x \in X\} \setminus U$. But any subsequence of $\{(x_{n_k}\}$ must converge to x, and so any subsequence of $\{(x_{n_k}, f(x_{n_k}))\}$ must converge to (x, y) for some y, which contradicts the fact that there is no point of the form (x, y) in $\{(x, f(x)) : x \in X\} \setminus U$. Hence it must be the case that $\{(x_n, f(x_n))\}$ is eventually inside U, and so $\lim_{n\to\infty}(x_n, f(x_n)) = (x, f(x))$. By Theorem 37.2, this means $\lim_{n\to\infty} f(x_n) = f(x)$ and so f is continuous by Theorem 40.2.