Homework 7 Solutions

Math 171, Spring 2010
Henry Adams

42.1. Prove that none of the spaces \( \mathbb{R}^n, l^1, l^2, c_0, \) or \( l^\infty \) is compact.

\textit{Solution.} Let \( X = \mathbb{R}^n, l^1, l^2, c_0, \) or \( l^\infty \). Let \( 0 = (0, \ldots, 0) \) in the case \( X = \mathbb{R}^n \) and let \( 0 = (0, 0, \ldots, 0) \) in the case \( X = l^1, l^2, c_0, \) or \( l^\infty \). For \( n \in \mathbb{N} \), let \( B_n(0) \) be the ball of radius \( n \) about \( 0 \) with respect to the relevant metric on \( X \). Note that \( \mathcal{U} = \{ B_n(0) : n \in \mathbb{N} \} \) is an open cover of \( X \). However, if \( \mathcal{U} \) had a finite subcover, then we would have \( X = B_N(0) \) for some \( N \in \mathbb{N} \). This is a contradiction in all cases. In the case of \( X = \mathbb{R}^N \), note that \( (N + 1, 0, \ldots, 0) \notin B_N(0) \). In the case \( X = l^1, l^2, c_0, \) or \( l^\infty \), note that \( (N + 1, 0, \ldots) \notin B_N(0) \). Hence none of the spaces \( \mathbb{R}^n, l^1, l^2, c_0, \) or \( l^\infty \) is compact.

42.3. Let \( X_1, \ldots, X_n \) be a finite collection of compact subsets of a metric space \( M \). Prove that \( X_1 \cup X_2 \cup \cdots \cup X_n \) is a compact metric space. Show (by example) that this result does not generalize to infinite unions.

\textit{Solution.} Let \( \mathcal{U} \) be an open cover of \( X_1 \cup X_2 \cup \cdots \cup X_n \). Then \( \mathcal{U} \) is an open cover of \( X_i \) for each \( 1 \leq i \leq n \). Since each \( X_i \) is compact, there is a finite subcover \( U_i^* \) of \( X_i \) for each \( i \). Let \( U^* = \bigcup_{i=1}^n U_i^* \). Then \( U^* \) is finite as it is a finite union of finite collections. Also, \( U^* \) covers \( X_i \) for all \( i \) as \( U_i^* \) covers \( X_i \) for each \( i \). Therefore \( U^* \) is a finite subcollection of \( \mathcal{U} \) covering \( X_1 \cup X_2 \cup \cdots \cup X_n \), and so \( X_1 \cup X_2 \cup \cdots \cup X_n \) is compact.

To see that this result does not generalize to infinite unions, let \( M = \mathbb{R} \) and let \( X_n = [n-1, n] \) for all \( n \in \mathbb{N} \). Then each \( X_n \) is compact, but \( \bigcup_{n=1}^\infty X_n = \bigcup_{n=1}^\infty [n-1, n] = [0, \infty) \) is not compact.

42.5. A collection \( \mathcal{C} \) of subsets of a set \( X \) is said to have the finite intersection property if whenever \( \{ C_1, \ldots, C_n \} \) is a finite subcollection of \( \mathcal{C} \), we have \( C_1 \cap C_2 \cap \cdots \cap C_n \neq \emptyset \). Prove that a metric space \( M \) is compact if and only if whenever \( \mathcal{C} \) is a collection of closed subsets of \( M \) having the finite intersection property, we have \( \cap \mathcal{C} \neq \emptyset \).

\textit{Solution.} First, suppose that \( M \) is compact. Let \( \mathcal{C} \) be a collection of closed subsets of \( M \) having the finite intersection property. Let \( \mathcal{U} = \{ C^c : C \in \mathcal{C} \} \). Then \( \mathcal{U} \) is an open collection of open sets. Suppose for a contradiction that \( \cup \mathcal{U} = M \). Then since \( M \) is compact, there exists some finite subcover \( \mathcal{U}^* \) of \( \mathcal{U} \). Label the sets in \( \mathcal{U}^* \) as \( U_i^* = \{ C_1^c, \ldots, C_n^c \} \) with \( C_i \in \mathcal{C} \) for all \( i \). Since \( \mathcal{C} \) has the finite intersection property, we have \( C_1 \cap C_2 \cap \cdots \cap C_n \neq \emptyset \). Taking complements, we get \( C_1^c \cup C_2^c \cup \cdots \cup C_n \neq M \), contradicting the fact that \( \mathcal{U}^* \) is a cover of \( M \). Hence it must be that \( \cup \mathcal{U} \neq M \), and taking complements gives \( \cap \mathcal{C} \neq \emptyset \).

Next, suppose whenever \( \mathcal{C} \) is a collection of closed subsets of \( M \) having the finite intersection property, we have \( \cap \mathcal{C} \neq \emptyset \). Let \( \mathcal{U} \) be an open cover of \( M \). Let \( \mathcal{C} = \{ U^c : U \in \mathcal{U} \} \), so \( \mathcal{C} \) is a collection of closed subsets. Since \( \mathcal{U} \) is an open cover, we have \( \cup \mathcal{U} = M \) hence \( \cap \mathcal{C} = \emptyset \). By assumption, this means that \( U_1^c \cap \cdots \cap U_n^c = \emptyset \) for some finite subset of \( \mathcal{C} \). Taking complements, we get that \( U_1 \cup \cdots \cup U_n = M \) for some finite subset of \( \mathcal{U} \). This shows that \( M \) is compact.

42.10. Let \( \{ X_n \} \) be a sequence of compact subsets of a metric space \( M \) with \( X_1 \supset X_2 \supset X_3 \supset \cdots \). Prove that if \( U \) is an open set containing \( \bigcap X_n \), then there exists \( X_n \subset U \).

\textit{Solution.} Note \( M = U \cup \bigcup_{n=1}^\infty X_n^c \) so \( X_1 \subset (U \cup \bigcup_{n=2}^\infty X_n^c) \). Hence \( U, X_1^c, X_2^c, X_3^c, X_4^c, \ldots \) is an open cover of the compact space \( X_1 \). By definition of compactness, there exists some finite
42.12. A contractive mapping on \( M \) is a function \( f \) from the metric space \( (M,d) \) into itself satisfying 
\[
d(f(x), f(y)) < d(x,y) \quad \text{whenever } x, y \in M \text{ with } x \neq y.
\]
Prove that if \( f \) is a contractive mapping on a compact metric space \( M \), there exists a unique point \( x \in M \) with \( f(x) = x \).

**Solution.** Suppose there does not exist such a fixed point \( x \) with \( f(x) = x \). Then the function \( g(x) = d(f(x), x) \) positive. To see that \( g \) is continuous, use the fact that \( f \) is continuous (which follows since \( f \) satisfies the contractive mapping property) and the triangle inequality. Since \( M \) is compact, by Corollary 42.7 there exists \( c \in M \) such that \( g(c) \leq g(x) \) for all \( x \in M \). As \( g \) is positive, this means that \( g(c) > 0 \). However, note

\[
g(f(c)) = d(f(f(c)), f(c)) < d(f(c), c) = g(c).
\]

This is a contradiction. Therefore, there must exist a fixed point \( x \in M \) with \( f(x) = x \). This fixed point must be unique, for if there were some \( y \neq x \) with \( f(y) = y \), then we would have \( d(f(x), f(y)) = d(x, y) \), which contradicts our hypotheses.

43.1. Prove that the set \( \{x \in M : d(x, 0) = 1\} \) is closed and bounded in \( M \), but not compact if \( M \) is \( l^2 \), \( c_0 \), or \( l^\infty \).

**Solution.** Let \( f(x) = d(x, 0) \), which is a continuous function by Theorem 40.3. So \( \{x \in M : d(x, 0) = 1\} \) is the continuous preimage of a closed set, hence closed by Theorem 40.5(ii).

Note that \( d(y, z) \leq 2 \) for all \( y, z \in \{x \in M : d(x, 0) = 1\} \), as \( d(y, z) \leq d(y, 0) + d(0, z) = 1 + 1 = 2 \). Hence \( \{x \in M : d(x, 0) = 1\} \) is bounded by Definition 43.6.

Let \( \delta^{(k)} \) in \( l^2 \), \( c_0 \), or \( l^\infty \) be given by

\[
\delta^{(k)}_n = \begin{cases} 
1 & \text{if } n = k \\
0 & \text{if } n \neq k. 
\end{cases}
\]

Check that \( \{\delta^{(k)}\}_{k=1}^\infty \) is a sequence of points in \( l^2 \), \( c_0 \), or \( l^\infty \) that has no convergent subsequence. Therefore \( l^2 \), \( c_0 \), and \( l^\infty \) are not compact by Theorem 43.5.

43.4. If \( (M,d) \) is a bounded metric space, we let \( \text{diam} \ M = \text{lub}\{d(x, y) : x, y \in M\} \). Prove that if \( (M,d) \) is a compact metric space, there exist \( x, y \in M \) such that \( d(x, y) = \text{diam} \ M \).

**Solution.** I will give two solutions.

First solution: For each \( x \in M \), define \( f(x) = \max\{d(x, y) : y \in M\} \), where this maximum is realized by Theorem 40.3 and Corollary 42.7. Let \( y_x \in M \) be a point such that \( f(x) = d(x, y_x) \).

We want to show that \( f \) is continuous. Let \( \epsilon > 0 \). Suppose \( d(x, x') < \epsilon \). It must be that \( d(x, y_x) < \epsilon + d(x', y_x) \), for otherwise we would have

\[
d(x, y_x) \leq d(x, x') + d(x', y_x) < \epsilon + d(x', y_x) \leq d(x, y_x),
\]

contradicting the choice of \( y_x \). Similarly, it must be that \( d(x', y_{x'}) < \epsilon + d(x, y_x) \). Together, these two inequalities show that \( |d(x, y_x) - d(x', y_{x'})| < \epsilon \). Hence

\[
|f(x) - f(x')| = |d(x, y_x) - d(x', y_{x'})| < \epsilon.
\]

This shows \( f \) is continuous.

Therefore, we apply Corollary 42.7 to see that there exists some \( c \in M \) such that \( f(c) \geq f(x) \) for all \( x \in M \). Hence

\[
d(c, y_c) = f(c) \geq f(x) = \max\{d(x, y) : y \in M\}
\]
44.6(a,b,c). Let $44.1$. Give an example of metric spaces $X$

43.7. Let $M$ be a compact subset of a metric space $M$. If $y \in X^c$, prove that there exists a point $a \in X$ such that $d(a, y) \leq d(x, y)$ for all $x \in X$. Give an example to show that the conclusion may fail if “compact” is replaced by “closed.”

\textbf{Solution.} Let $f(x) = d(x, y)$. By Theorem 40.3, the function $f$ is continuous. By Corollary 42.7, there exists a point $a \in X$ such that $d(a, y) \leq d(x, y)$ for all $x \in X$.

To see that the conclusion may fail if “compact” is replaced by “closed,” let $M = l^\infty$. Let $\delta^{(k)} \in l^\infty$ be given by

$$\delta^{(k)} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k, \end{cases}$$

and let $X = \{\delta^{(k)} : k \in \mathbb{P}\}$. Note that $X$ contains its limit points and is therefore closed. Let $y = (-1, -\frac{1}{2}, \ldots, -\frac{1}{n}, \ldots)$. So $d(\delta^{(k)}, y) = 1 + \frac{1}{k}$ for all $k \in \mathbb{P}$, which implies that there does not exist a fixed $k_0 \in \mathbb{P}$ such that $d(\delta^{(k_0)}, y) \leq d(\delta^{(k)}, y)$ for all $k \in \mathbb{P}$.

44.1. Give an example of metric spaces $M_1$ and $M_2$ and a continuous function $f$ from $M_1$ onto $M_2$ such that $M_2$ is compact, but $M_1$ is not compact.

\textbf{Solution.} Let $M_1 = \mathbb{R}$, let $M_2$ be the trivial metric space $\{0\}$ consisting of a single point, and let $f : \mathbb{R} \to \{0\}$ be given by $f(x) = 0$ for all $x \in \mathbb{R}$. Check that $f$ is a continuous function. Note that $M_2 = \{0\}$ is compact, but $M_1 = \mathbb{R}$ is not compact.

44.6(a,b,c). Let $f$ be a one-to-one function from a metric space $M_1$ onto a metric space $M_2$. If $f$ and $f^{-1}$ are continuous, we say that $f$ is a homeomorphism and that $M_1$ and $M_2$ are homeomorphic metric spaces.

(a) Prove that any two closed intervals of $\mathbb{R}$ are homeomorphic.

\textbf{Solution.} Let $[a, b]$ and $[c, d]$ be any two closed intervals of $\mathbb{R}$. Define $f : [a, b] \to [c, d]$ by $f(x) = \frac{d-c}{b-a}(x-a) + c$. Check that $f$ is one-to-one and onto, and that $f^{-1}[c, d] \to [a, b]$ is given by $f^{-1}(x) = \frac{b-d}{a-c}(x-c) + a$. Check that $f$ and $f^{-1}$ are continuous functions, and hence $[a, b]$ and $[c, d]$ are homeomorphic.

(b) Prove (a) with “closed” replaced by “open”; with “closed” replaced by “half-open”.

\textbf{Solution.} When “closed” is replaced by “open”, the argument given in (a) works after replacing $[a, b]$ and $[c, d]$ with $(a, b)$ and $(c, d)$, respectively.

When “closed” is replaced by “half-open,” there are four cases. If the two intervals are $(a, b]$ and $(c, d]$ or $(a, b]$ and $[c, d)$, then define $f$ by $f(x) = -\frac{d-c}{b-a}(x-a) + d$, and proceed as above.

(c) Prove that a closed interval is not homeomorphic to either an open interval or a half-open interval.

\textbf{Solution.} I will prove the following claim: if two spaces $M_1$ and $M_2$ are homeomorphic, then $M_1$ is compact if and only if $M_2$ is compact. For the proof, note that if $M_1$ is compact, then $M_2 = f(M_1)$

for all $x \in M$. This shows that $d(c, y_c) = \text{lub}\{d(x, y) : x, y \in M\} = \text{diam } M$.

Second solution: Consider the product metric space $(M \times M, d')$, where $d'$ is defined by $d'[(x_1, x_2), (y_1, y_2)] = d(x_1, y_1) + d(y_1, y_2)$ as in Exercise 35.8. Since $(M, d)$ is compact, by Exercise 43.2 it follows that $(M \times M, d')$ is compact. Show that $d : M \times M \to \mathbb{R}$ is continuous, using the definition of $d'$ and the triangle inequality. So Corollary 42.7 tells us that there exist points $(c, d) \in M \times M$ such that $d(c, d) \geq d(x, y)$ for all $x, y \in M$. Hence $d(c, d) = \text{diam } M$. 

43.7. Let $M$ be a compact subset of a metric space $M$. If $y \in X^c$, prove that there exists a point $a \in X$ such that $d(a, y) \leq d(x, y)$ for all $x \in X$. Give an example to show that the conclusion may fail if “compact” is replaced by “closed.”

\textbf{Solution.} Let $f(x) = d(x, y)$. By Theorem 40.3, the function $f$ is continuous. By Corollary 42.7, there exists a point $a \in X$ such that $d(a, y) \leq d(x, y)$ for all $x \in X$.

To see that the conclusion may fail if “compact” is replaced by “closed,” let $M = l^\infty$. Let $\delta^{(k)} \in l^\infty$ be given by

$$\delta^{(k)} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k, \end{cases}$$

and let $X = \{\delta^{(k)} : k \in \mathbb{P}\}$. Note that $X$ contains its limit points and is therefore closed. Let $y = (-1, -\frac{1}{2}, \ldots, -\frac{1}{n}, \ldots)$. So $d(\delta^{(k)}, y) = 1 + \frac{1}{k}$ for all $k \in \mathbb{P}$, which implies that there does not exist a fixed $k_0 \in \mathbb{P}$ such that $d(\delta^{(k_0)}, y) \leq d(\delta^{(k)}, y)$ for all $k \in \mathbb{P}$.
is compact by Theorem 44.1. Conversely, if $M_2$ is compact, then $M_1 = f^{-1}(M_2)$ is compact by Theorem 44.1.

Since a closed interval is compact but an open interval or a half-open interval is not compact, our claim shows that a closed interval is not homeomorphic to either an open interval or a half-open interval.

44.8. Let $X$ be a compact subset of $\mathbb{R}$, and let $f$ be a real-valued function on $X$. Prove that $f$ is continuous if and only if $\{(x, f(x)) : x \in X\}$ is a compact subset of $\mathbb{R}^2$.

Solution. First, suppose $f$ is continuous. Let $\{(x_n, f(x_n))\}$ be any sequence of points in $\{(x, f(x)) : x \in X\}$. Then $\{x_n\}$ is a sequence of points in $X$. Since $X$ is compact, by Theorem 43.5 there is a convergent subsequence $\{x_{n_k}\}$ with $\lim_{k \to \infty} x_{n_k} = x \in X$. Since $f$ is continuous, by Theorem 40.2 we have $\lim_{k \to \infty} f(x_{n_k}) = f(x)$. Hence $\lim_{k \to \infty} (x_{n_k}, f(x_{n_k})) = (x, f(x))$ by Theorem 37.2. Hence $\{(x, f(x)) : x \in X\}$ is compact by Theorem 43.5.

Conversely, suppose that $\{(x, f(x)) : x \in X\}$ is a compact subset of $\mathbb{R}^2$. Let $\{x_n\}$ be any sequence of points in $X$ with $\lim_{n \to \infty} x_n = x \in X$. Consider the sequence of points $\{(x_n, f(x_n))\}$ in $\{(x, f(x)) : x \in X\}$. Let $U$ be any open neighborhood about $(x, f(x))$ in $\mathbb{R}^2$. Suppose for a contradiction that $\{(x, f(x))\}$ is not eventually inside $U$. Then there is a subsequence $\{(x_{n_k}, f(x_{n_k}))\}$ which lies in $\{(x, f(x)) : x \in X\} \setminus U$. Note $\{(x, f(x)) : x \in X\} \setminus U$ is a closed subset of the compact set $\{(x, f(x)) : x \in X\}$ and is hence compact. So $\{(x_{n_k}, f(x_{n_k}))\}$ has a convergent subsequence in $\{(x, f(x)) : x \in X\} \setminus U$. But any subsequence of $\{x_{n_k}\}$ must converge to $x$, and so any subsequence of $\{(x_{n_k}, f(x_{n_k}))\}$ must converge to $(x, y)$ for some $y$, which contradicts the fact that there is no point of the form $(x, y)$ in $\{(x, f(x)) : x \in X\} \setminus U$. Hence it must be the case that $\{(x_n, f(x_n))\}$ is eventually inside $U$, and so $\lim_{n \to \infty} (x_n, f(x_n)) = (x, f(x))$. By Theorem 37.2, this means $\lim_{n \to \infty} f(x_n) = f(x)$ and so $f$ is continuous by Theorem 40.2.