Homework 6 Solutions

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38.6. Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Prove that $\{x : f(x) = 0\}$ is a closed subset of $\mathbb{R}$.

Solution. Let y be a limit point of $\{x : f(x) = 0\}$. So there is a sequence $\{y_n\}$ such that $y_n \in \{x : f(x) = 0\}$ for all n and $\lim_{n\to\infty} y_n = y$. Since f is continuous, by Theorem 40.2 we have $f(y) = \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 0 = 0$. Hence $y \in \{x : f(x) = 0\}$, so $\{x : f(x) = 0\}$ contains all of its limit points and is a closed subset of $\mathbb{R}$.

38.8. Let X and Y be closed subsets of $\mathbb{R}$. Prove that $X \times Y$ is a closed subset of $\mathbb{R}^2$. State and prove a generalization to $\mathbb{R}^n$.

Solution. The generalization to $\mathbb{R}^n$ is that if $X_1, \ldots, X_n$ are closed subsets of $\mathbb{R}$, then $X_1 \times \cdots \times X_n$ is a closed subset of $\mathbb{R}^n$. We prove this generalized statement, which in particular proves the case $n = 2$.

Let $(x_1, \ldots, x_n)$ be a limit point of $X_1 \times \cdots \times X_n$. So there exists a sequence $(x_1^{(k)}, \ldots, x_n^{(k)})$ in $X_1 \times \cdots \times X_n$ which converges to $(x_1, \ldots, x_n)$. By Theorem 37.2, we have $\lim_{k\to\infty} x_j^{(k)} = x_j$ for each $j = 1, \ldots, n$. Hence x_j is a limit point of X_j for each j. Since each X_j is closed, we have $x_j \in X_j$ for each j. Hence $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$.

38.9. Let (M, d) be a metric space. Let $\epsilon > 0$ and let $y \in M$. Prove that the closed ball $\{x : d(x, y) \leq \epsilon\}$ is a closed subset of M.

Solution. Let z be a limit point of $\{x : d(x, y) \leq \epsilon\}$. So there is a sequence $\{z_n\}$ such that $z_n \in \{x : d(x, y) \leq \epsilon\}$ for all n and $\lim_{n\to\infty} z_n = z$.

Suppose for a contradiction that $z \notin \{x : d(x, y) \leq \epsilon\}$. So $d(z, y) = \delta + \epsilon$ for some $\delta > 0$. Pick N such that $d(z, z_N) \leq \frac{\delta}{2}$. By the triangle inequality, we have

$$d(z, y) \leq d(z, z_N) + d(z_N, y) \leq \frac{\delta}{2} + \epsilon < \delta + \epsilon = d(z, y)$$

which is a contradiction. Hence it must be the case that $z \in \{x : d(x, y) \leq \epsilon\}$, so $\{x : d(x, y) \leq \epsilon\}$ contains all of its limit points and is a closed subset of M.

38.14. Let $\{x_n\}$ be a sequence in a metric space M with no convergent subsequence. Prove that $\{x_n : n \in \mathbb{P}\}$ is a closed subset of M.

Solution. Let z be a limit point of $\{x_n : n \in \mathbb{P}\}$. So there is a sequence $\{z_k\}$ such that $x_k \in \{x_n : n \in \mathbb{P}\}$ for all k and $\lim_{k\to\infty} z_k = z$.

Suppose for a contradiction that $z \notin \{x_n : n \in \mathbb{P}\}$. By induction on m, we define a sequence $\{a_m\}$ which is a subsequence of both $\{x_n\}$ and $\{z_k\}$. For the base case, set $a_1 = z_1 = x_n$ for some integer n. For the inductive step, suppose we have defined $a_1, \ldots, a_m$ and $a_m = z_k = x_n$. Note the set $\{z_{k+1}, z_{k+2}, \ldots\}$ is infinite for otherwise some x_j appears in this set an infinite number of times, contradicting the fact that $\lim_{k\to\infty} z_k = z \neq x_j$. Since $x_1, x_2, \ldots$ is an enumeration of $\{x_n : n \in \mathbb{P}\}$, and since the set $\{z_{k+1}, z_{k+2}, \ldots\}$ is infinite but $\{x_1, \ldots x_n\}$ is finite, there exists some $n' > n$ such that $x_{n'} = z_{k'}$ for some $k' > k$. Set $a_{m+1} = z_{k'} = x_{n'}$. Note that $\{a_m\}$ is a subsequence of both

$\{z_k\}$ and $\{x_n\}$. Since $\{z_k\}$ converges, so does $\{a_m\}$, contradicting the assumption that $\{x_n\}$ has no convergent subsequence.

(Note the similarities with the solution of Exercise 20.7 from Homework 3).

39.4. Let M be a metric space such that M is a finite set. Prove that every subset of M is open.

Solution. Let X be a subset of M. Since M is finite, the complement X' is finite. By Corollary 38.7, X' is closed. By Theorem 39.5, X is open. Hence every subset of M is open.

39.5. Prove that the interior of a rectangle in $\mathbb{R}^2$

$$\{(x, y) : a < x < b, c < y < d\}$$

is an open subset of $\mathbb{R}$.

Solution. Suppose (x', y') satisfies $a < x' < b$ and $c < y' < d$. Let $\epsilon = \min\{x'-a, b-x', y'-c, d-y'\}$. Then $B_\epsilon((x', y')) \subset \{(x, y) : a < x < b, c < y < d\}$. This is easy to see, since if $d((x', y'), (x, y)) < \epsilon$, then necessarily $a \leq x' - \epsilon < x < x' + \epsilon \leq b$ and $c \leq y' - \epsilon < y < y' + \epsilon \leq d$.

39.7. Let f be a continuous function from $\mathbb{R}$ into $\mathbb{R}$. Prove that $\{x : f(x) > 0\}$ is an open subset of $\mathbb{R}$.

Solution. Suppose $y \in \{x : f(x) > 0\}$. So $f(y) > 0$. By Theorem 33.3, since f is continuous, there exists some δ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < f(y)$, which implies $-f(y) < f(x) - f(y)$ and hence $0 < f(x)$. That is, if $x \in B_\delta(y)$, then $f(x) > 0$. So $\{x : f(x) > 0\}$ is an open subset of $\mathbb{R}$ by Definition 39.2.

39.9. Let X be a subset of a metric space M. Prove that X is an open subset of M if and only if X is the union of open balls.

Solution. First, suppose X is an open subset of M. Then by Definition 39.2, for every $x \in X$ there exists an open ball $B_{\epsilon_x}(x)$, where ϵ_x depends on x, such that $B_{\epsilon_x} \subset X$. Note that

$$X \subset \cup_{x \in X} B_{\epsilon_x}(x) \subset X.$$

So $X = \cup_{x \in X} B_{\epsilon_x}(x)$ is the union of open balls.

Conversely, suppose X is the union of open balls. Each ball is open by Theorem 39.4, and so as the union of open sets, X is open by Theorem 39.6(ii).

40.8. Let f and g be continuous functions from $\mathbb{R}$ into $\mathbb{R}$. Prove that $h(x) = (f(x), g(x))$ defines a continuous function from $\mathbb{R}$ into $\mathbb{R}^2$. State and prove generalizations involving continuous functions from $\mathbb{R}^m$ into $\mathbb{R}^n$.

Solution. The generalized statement is that if $f_1, \ldots, f_n$ are continuous functions from $\mathbb{R}^m$ into $\mathbb{R}$, then $h(x) = (f_1(x), \ldots, f_n(x))$ defines a continuous function from $\mathbb{R}^m$ into $\mathbb{R}^n$. We prove this generalized statement, which in particular proves the case $m = 1$ and $n = 2$.

Let $a \in \mathbb{R}^m$ and $\epsilon > 0$. Since each f_i is continuous for $i = 1, \ldots, n$, by Definition 40.1 there exists δ_i such that if $d(x, a) < \delta_i$, then $|f_i(x) - f_i(a)| < \sqrt{\epsilon^2/n}$. Let $\delta = \min\{d_1, \ldots, \delta_i\}$. So if $d(x, a) < \delta$, then $|f_i(x) - f_i(a)| < \sqrt{\epsilon^2/n}$ for all i, and so

$$d(h(x), h(a)) = \sqrt{\sum_{i=1}^{n} |f_i(x) - f_i(a)|^2} \leq \sqrt{\sum_{i=1}^{n} \epsilon^2/n} = \sqrt{\epsilon^2} = \epsilon$$

Hence h is a continuous function from $\mathbb{R}^m$ into $\mathbb{R}^n$.

40.10. Let $\{a_n\} \in l^\infty$. Prove that f defined by $f(\{b_n\}) = \sum_{n=1}^{\infty} a_n b_n$ is a continuous real-valued function on l^1.

Solution. Let $\{b_n\} \in l^1$ and $\epsilon > 0$. Since $\{a_n\} \in l^\infty$ there exists some M such that $|a_n| < M$ for all n. Let $\delta = \frac{\epsilon}{M}$. If $\{c_n\} \in l^1$ and $d(\{c_n\}, \{b_n\}) < \delta$, then $\sum_{n=1}^{\infty} |c_n - b_n| < \delta = \frac{\epsilon}{M}$. So

$$|f(\{c_n\}) - f(\{b_n\})| = |\sum_{n=1}^{\infty} a_n c_n - \sum_{n=1}^{\infty} a_n b_n| = |\sum_{n=1}^{\infty} a_n (c_n - b_n)| \leq M \sum_{n=1}^{\infty} |c_n - b_n| \leq M \frac{\epsilon}{M} = \epsilon.$$

Hence f is continuous by Definition 40.1.

40.15. Let f be a real-valued function on a metric space M. Prove that f is continuous on M if and only if the sets $\{x : f(x) < c\}$ and $\{x : f(x) > c\}$ are open in M for every $c \in \mathbb{R}$.

Solution. First suppose that f is continuous. Note that $(-\infty, c)$ and (c, ∞) are open subsets of $\mathbb{R}$. Hence $\{x : f(x) < c\} = f^{-1}((-\infty, c))$ and $\{x : f(x) > c\} = f^{-1}((c, \infty))$ are open in M by Theorem 40.5(iii).

Conversely, suppose the sets $\{x : f(x) < c\}$ and $\{x : f(x) > c\}$ are open in M for every $c \in \mathbb{R}$. By Exercise 39.9, any open subset U of $\mathbb{R}$ can be written as the union of open balls $U = \cup_{\alpha \in A} (a_\alpha, b_\alpha)$, where A is an arbitrary indexing set. Note $(a_\alpha, b_\alpha) = (-\infty, b_\alpha) \cap (a_\alpha, \infty)$ and

$$f^{-1}((a_\alpha, b_\alpha)) = f^{-1}((-\infty, b_\alpha)) \cap f^{-1}((a_\alpha, \infty)) = \{x : f(x) < b_\alpha\} \cap \{x : f(x) > a_\alpha\}.$$

Since the intersection of any two open sets is open, each set $f^{-1}((a_\alpha, b_\alpha))$ is open. Since the arbitrary union of open sets is open, the set $f^{-1}(U) = \cup_{\alpha \in A} f^{-1}((a_\alpha, b_\alpha))$ is open. Hence by Theorem 40.5(iii), f is continuous.

40.17. Let M be a set and let d and d' be metrics for M. We say that d and d' are equivalent metrics for M if the collection of open subsets of (M, d) is identical with the collection of open subsets of (M, d').

(a) Prove that the following are equivalent.
(i) d and d' are equivalent metrics.
(ii) The collection of closed subset of (M, d) is identical with the collection of closed subsets of (M, d').
(iii) The sequence $\{x_n\}$ converges in (M, d) if and only if $\{x_n\}$ converges in (M, d').

Solution. We show (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

Suppose (i) is true, and suppose the sequence $\{x_n\}$ converges to x in (M, d). We want to show that $\{x_n\}$ converges in (M, d'). Let $\epsilon > 0$. Let $B_\epsilon^{d'}(x)$ be the ball radius ϵ about x in (M, d'). As $B_\epsilon^{d'}(x)$ is open in (M, d'), by (i) it is open in (M, d). Hence there exists some $\delta > 0$ such that $B_\delta^d(x) \subset B_\epsilon^{d'}(x)$. Since $\{x_n\}$ converges to x in (M, d), there exists some N such that if $n \geq N$ then $x_n \in B_\delta^d(x)$ for all $n \geq N$. This shows $x_n \in B_\epsilon^{d'}(x)$ for all $n \geq N$. Hence $\{x_n\}$ converges to x in (M, d'). Similarly, we can use (i) to show that if $\{x_n\}$ converges in (M, d') then $\{x_n\}$ converges in (M, d). So (i) $\Rightarrow$ (iii).

Note that statement (iii) implies the (a priori stronger) statement that the sequence $\{x_n\}$ converges to x in (M, d) if and only if $\{x_n\}$ converges to x in (M, d'). To see this, consider the interleaved sequence

$$x_1, x, x_2, x, x_3, x, x_4, x, \ldots$$

Note $\{x_n\}$ converges to x in (M, d) if and only if the interleaved sequence converges in (M, d) if and only if the interleaved sequence converges in (M, d') (by (iii)) if and only if the interleaved sequence converges in (M, d') if and only if $\{x_n\}$ converges to x in (M, d'). Hence x is a limit point of a set X in (M, d) if and only if x is a limit point of X in (M, d'), and so X is closed in (M, d) if and only if X is closed in (M, d'). This shows (iii) $\Rightarrow$ (ii).

By Theorem 39.5, a set in a metric space is open if and only if its complement is closed. So the collection of open subsets are identical if and only if the collection of closed subsets are identical, giving (ii) $\Rightarrow$ (i).

(b) Prove that the metrics d, d', and d'' of Exercise 35.7 are equivalent.

Solution. By Exercise 37.10, $\{a_n\}$ converges to a in $(M, d) \Leftrightarrow \{a_n\}$ converges to a in $(M, d') \Leftrightarrow \{a_n\}$ converges to a in (M, d''). Hence the metrics d, d', and d'' are equivalent by condition (iii) above.

(c) Prove that the metric d' of Exercise 37.9 is equivalent to the ususal (Euclidean) metric on $\mathbb{R}^n$.

Solution. By Exercise 37.9(b), a sequence of points $\{a^{(k)}\}$ converges to a in $(\mathbb{R}^n, d')$ if and only if $\{a^{(k)}\}$ converges to a in $(\mathbb{R}^n, d)$, where d is the usual metric. Hence the metrics d and d' are equivalent by condition (iii) above.

41.3. Let $A = [0, 1]$, $B = (\frac{1}{2}, 1]$, and $C = (\frac{1}{4}, \frac{3}{4})$.

Solution.
(a) is B open (closed) in A?
By Theorem 41.2, B is open in A but not closed in A.
(b) is C open (closed) in A?
By Theorem 41.2, C is open in A but not closed in A.
(c) is A open (closed) in $\mathbb{R}$?
A is closed but not open in $\mathbb{R}$.
(d) is C open (closed) in $\mathbb{R}$?
C is open but not closed in $\mathbb{R}$.
(e) is A open (closed) in $\mathbb{R}^2$?
By Definition 39.2, A is not open in $\mathbb{R}^2$. To see that A is closed in $\mathbb{R}^2$, note that $\mathbb{R}$ is closed in $\mathbb{R}^2$ by (g) and that A is closed in $\mathbb{R}$ by (c). By Corollary 41.3, A is closed in $\mathbb{R}^2$.
(f) is C open (closed) in $\mathbb{R}^2$?
By Definition 39.2, C is not open in $\mathbb{R}^2$. Since C is not closed in $\mathbb{R}$ and $\mathbb{R}$ is closed in $\mathbb{R}^2$ by (g), Corollary 41.3 tells us that C is not closed in $\mathbb{R}^2$.
(g) is $\mathbb{R}$ open (closed) in $\mathbb{R}^2$?
The book is thinking of $\mathbb{R} \subset \mathbb{R}^2$ as $\mathbb{R} = \{(x, y) : y = 0\}$. By Definition 39.2, $\mathbb{R}$ is not open in $\mathbb{R}^2$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f((x, y)) = y$. Note that f is continuous and that $\mathbb{R} = f^{-1}(\{0\})$. Hence $\mathbb{R}$ is a closed subset of $\mathbb{R}^2$ by Theorem 40.5(ii).

41.4. Let M be a metric space and let X be a subset of M with the relative metric. Prove that if f is a continuous function on M, then $f|_X$ is a continuous function on X.

Solution. Since f is a continuous function on M, by Theorem 40.5 $f^{-1}(U)$ is open for all open sets U in the range of f. Note that $f|_X^{-1}(U) = f^{-1}(U) \cap X$ is open in X by Theorem 41.2. Hence $f|_X$ is a continuous function on X by Theorem 40.5.