SOLUTIONS FOR PROBLEM SET 8

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1. By partial summation we have

$$\sum_{n=1}^{\infty}\mu(n)\Big(\frac{1}{n}-\frac{1}{n^{\sigma}}\Big)=\int_{1^-}^{\infty}\Big(\frac{1}{y}-\frac{1}{y^{\sigma}}\Big)d\Big(\sum_{n\le y}\mu(n)\Big),$$

and integrating by parts this equals

$$\int_{1^-}^{\infty}\Big(\sum_{n\le y}\mu(n)\Big)\Big(\frac{1}{y^2}-\frac{\sigma}{y^{\sigma+1}}\Big)dy.$$

Now for $\sigma>1$ we have that $1/y^2-\sigma/y^{\sigma+1}=O((\sigma-1)/y^2)$, and by our proof of the prime number theorem we have that $\sum_{n\le y}\mu(n)\ll y/(\log 2y)^A$ for any A (the $\log 2y$ instead of $\log y$ is just to take care of the technical point when y is close to 1). Hence

$$\sum_{n=1}^{\infty}\mu(n)\Big(\frac{1}{n}-\frac{1}{n^{\sigma}}\Big)\ll(\sigma-1)\int_{1}^{\infty}\frac{1}{y(\log y)^A}dt\ll(\sigma-1),$$

by choosing $A=2$, say. This gives our result.

2. We will use induction on k as suggested. The case $k=1$ is the prime number theorem.

Let $n\le x$ have exactly k-prime factors ($\Omega(n)=k$) and let P denote the largest prime factor of n. We distinguish two cases based on whether $P>\sqrt{x}$, or not. The main contribution will come from the first case. Let us first dispose of the second case, showing that it contributes a negligible amount.

Note that if $P\le x^{1/(2k)}$ then $n\le x^{1/2}$ and the number of such n is at most $O(x^{1/2})$ which is negligible. Now suppose $x^{1/(2k)}\le P\le\sqrt{x}$. Then $n/P\le x/P$ has $k-1$-prime factors, all at most P, and so by induction hypothesis the number of choices for n/P is

$$\ll\frac{x}{P\log(x/P)}(\log\log x/P)^{k-2}\ll\frac{x}{P\log x}(\log\log x)^{k-2}.$$

Summing this over $x^{1/(2k)} \le P \le \sqrt{x}$ we see that the number of such integers is

$$\ll \frac{x}{\log x}(\log\log x)^{k-2} \sum_{x^{1/(2k)} \le P \le \sqrt{x}} \frac{1}{P} \ll \frac{x}{\log x}(\log\log x)^{k-2}.$$

Therefore, the second case contributes a negligible amount, as claimed.

Now for the first case when $P > \sqrt{x}$. Here we are counting (with $n = Pm$)

$$\sum_{\sqrt{x} \le P \le x/2} \sum_{\substack{m \le x/P \\ \Omega(m)=k-1}} 1 \sim \sum_{\sqrt{x} \le P \le x/2} \frac{x}{P\log(x/P)} \frac{(\log\log(x/P))^{k-2}}{(k-2)!},$$

by induction hypothesis. At this stage, use the prime number theorem and partial summation to handle the sum over P. Suppressing some details, this gives that the above is

$$\sim x \int_{\sqrt{x}}^{x/2} \frac{dt}{t(\log t)(\log x/t)} \frac{(\log\log x/t)^{k-2}}{(k-2)!}$$

Making a change of variables $\log(x/t) = y$ so that $dt/t = -dy$ we get that the above is

$$\sim x \int_{\log 2}^{\log\sqrt{x}} \frac{dy}{y(\log x - y)} \frac{(\log y)^{k-2}}{(k-2)!} = x \int_{\log 2}^{\log\sqrt{x}} \frac{dy}{y\log x}\Big(1+O\Big(\frac{y}{\log x}\Big)\Big) \frac{(\log y)^{k-2}}{(k-2)!}.$$

The above equals

$$\frac{x}{\log x}\frac{(\log\log\sqrt{x})^{k-1}}{(k-1)!} + O\Big(\frac{x}{\log x}(\log\log x)^{k-2}\Big) \sim \frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!},$$

as desired. (Well, that was a little harder than I thought!)

3. Put $S(x) = S(x,\chi_{-4}) = \sum_{n \le x} \chi_{-4}(n)$. Note that $S(x)$ is periodic in x with period 4, and hence $|S(x)|$ is bounded; indeed bounded by 1. In the region $\sigma > 1$ we have that

$$L(s,\chi_{-4}) = \int_{1^-}^{\infty} \frac{1}{y^s} d(S(y)) = s\int_{1^-}^{\infty} \frac{S(y)}{y^{s+1}} dy.$$

Since $|S(y)|$ is bounded, the integral above converges absolutely in $\sigma > 0$. This furnishes an analytic continuation of $L(s,\chi_{-4})$ to this region.

To obtain bounds for $L(s,\chi_{-4})$, set $N = [|s|] + 2$ and note that

$$L(s,\chi_{-4}) = \sum_{n \le N} \frac{\chi_{-4}(n)}{n^s} + \int_{N^+}^{\infty} \frac{1}{y^s} dS(y) = \sum_{n \le N} \frac{\chi_{-4}(n)}{n^s} - \frac{S(N^+)}{N^s} + s\int_{N^+}^{\infty} \frac{S(y)}{y^{s+1}} dy.$$

Recalling that $|S(y)| \le 1$, we obtain that this is bounded in magnitude by (for $\sigma > 0$)

$$\ll \sum_{n \le N} \frac{1}{n^\sigma} + N^{-\sigma} + \frac{|s|}{\sigma N^\sigma}$$

and we may check that this is (the $\log(1+|s|)$ below is to account for what happens if $\sigma = 1$)

$$\ll \sigma^{-1}(1 + (1+|s|)^{1-\sigma}) \log(1+|s|).$$

4. When $t = 0$ we have $L(1, \chi_{-4}) = 1 - 1/3 + 1/5 - 1/7 + \ldots$ which we either see immediately as being positive, or recognize as being $\pi/4$. In either case it is not zero. Therefore we also know that there is a neighborhood $|s-1| \le \delta$ where $L(s, \chi_{-4}) \ne 0$. (The zeros of an analytic function are isolated.)

Now suppose that $|t| \ge \delta$ is not too close to zero, and we want to prove that $L(1+it, \chi_{-4}) \ne 0$. For $\sigma > 1$ we look at

$$\zeta(\sigma)^3 |L(\sigma + it, \chi_{-4})|^4 |\zeta(\sigma + 2it)|$$

and claim that it is ≥ 1. To see this take the logarithm of the LHS. Expanding that into its Dirichlet series we obtain

$$\sum_n \frac{\Lambda(n)}{n^\sigma \log n}\Big(3 + 4\chi_{-4}(n)\cos(t\log n) + \cos(2t\log n)\Big).$$

If n is a power of 2 then $\chi_{-4}(n) = 0$ and the term in brackets above is $3 + \cos(2t\log n) \ge 0$. For other n, $\chi_{-4}(n) = \pm 1$, and the term in brackets is $2(1 \pm \cos(t\log n))^2 \ge 0$. Hence our claim follows.

Now let $\sigma \to 1^+$. If $L(1+it, \chi_{-4}) = 0$ then $L(\sigma+it, \chi_{-4})$ would be approximately $C(\sigma - 1)$ for some constant C (as $\sigma \to 1^+$). Moreover we know that $\zeta(\sigma) \sim 1/(\sigma-1)$, and $\zeta(\sigma+2it) \to \zeta(1+2it)$ which (since t is not close to zero) is bounded. But then $\zeta(\sigma)^3|L(\sigma+it, \chi_{-4})|^4|\zeta(\sigma+2it)|$ would tend to $C^4(\sigma-1)|\zeta(1+2it)|$ which is not ≥ 1 for σ sufficiently close to 1. This contradiction shows that $L(1+it, \chi_{-4}) \ne 0$.

5. We follow the outline of the proof of the prime number theorem discussed in class. We consider

$$\begin{aligned} M_k(x, \chi_{-4}) &= \sum_{n \le x} \chi_{-4}(n)\mu(n)(\log n)^k \log(x/n) \\ &= \frac{(-1)^k}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Big(\frac{1}{L(s, \chi_{-4})}\Big)^{(k)} \frac{x^s}{s^2} ds, \end{aligned}$$

where we take $c = 1 + 1/\log x$.

We now need to bound $(1/L(s,\chi_{-4}))^{(k)}$. We may write this as an expression involving the j-th derivatives of $L(s,\chi_{-4})$ (with $0 \le j \le k$) divided by $L(s,\chi_{-4})^{k+1}$. The numerator is bounded above by $(\log(2+|t|))^{C(k)}$ for some constant $C(k)$, exactly as in the case of ζ discussed in class. The denominator is at least $(\log x)^{-\frac{3}{4}(k+1)}(\log(2+|t|))^{-\frac{(k+1)}{4}}$, by using our work in Problem 4 (when $|t| \ge \delta$, and for smaller t the denominator is $\gg 1$). So we find that $(1/L(s,\chi_{-4}))^{(k)}$ is bounded by $(\log x)^{\frac{3}{4}(k+1)}(\log(2+|t|))^{D(k)}$ for some constant $D(k)$. Using this above we find that

$$|M_k(x,\chi_{-4})| \ll x(\log x)^{\frac{3}{4}(k+1)} \int_{-\infty}^{\infty} (\log(2+|t|))^{D(k)} \frac{dt}{1+|t|^2} \ll x(\log x)^{\frac{3}{4}(k+1)}.$$

Now we unsmooth as in class: consider for $1 \le y \le x$

$$M_k(x+y,\chi_{-4}) - M_k(x,\chi_{-4}) = \log\frac{x+y}{x} \sum_{n \le x} \mu(n)\chi_{-4}(n)(\log n)^k + O\Big(\log\frac{x+y}{x} y(\log x)^k\Big),$$

so that using $\log(1+y/x)$ is approximately y/x and our estimate above, we obtain that

$$\sum_{n \le x} \mu(n)\chi_{-4}(n)(\log n)^k \ll \frac{x^2}{y}(\log x)^{\frac{3}{4}(k+1)} + y(\log x)^k.$$

Choosing $y = x(\log x)^{-\frac{k}{8}+\frac{3}{8}}$ we conclude that

$$\sum_{n \le x} \mu(n)\chi_{-4}(n)(\log n)^k \ll x(\log x)^{\frac{7}{8}k+\frac{3}{8}}.$$

Now using partial summation we conclude that

$$\sum_{n \le x} \mu(n)\chi_{-4}(n) \ll x(\log x)^{-\frac{1}{8}k+\frac{3}{8}},$$

and since k is arbitrary this is $\ll_A x/(\log x)^A$ for any $A > 0$.

Next note that

$$\sum_{n \le x} \mu(n) = \sum_{\substack{n \le x \\ n \text{ odd}}} \mu(n) + \sum_{\substack{n \le x/2 \\ n \text{ odd}}} \mu(2n) = \sum_{\substack{x/2 \le n \le x \\ n \text{ odd}}} \mu(n),$$

and so the RHS above is $\ll x/(\log x)^A$ for any $A > 0$. Summing this with x replaced by $x/2$, $x/4$, etc we conclude that

$$\sum_{\substack{n \le x \\ n \text{ odd}}} \mu(n) \ll \frac{x}{(\log x)^A}.$$

Adding to this estimate our estimate for $\sum_{n\le x}\mu(n)\chi_{-4}(n)$ we conclude that

$$\sum_{\substack{n\le x\\ n\equiv 1 \pmod 4}} \mu(n) \ll \frac{x}{(\log x)^A}.$$

Note that a similar estimate holds for $n \equiv 3 \pmod 4$. These estimates giving cancellation in $\mu(n)$ in progressions are equivalent to the prime number theorem in these arithmetic progressions.