1. Put $\delta_k(n)$ to be 1 if $n$ is $k$-free and 0 otherwise. Then $\delta_k(n)$ is a multiplicative function and

$$F(s) := \sum_{n=1}^{\infty} \frac{\delta_k(n)}{n^s} = \prod_p \left( 1 + \frac{1}{p^s} + \ldots + \frac{1}{p^{(k-1)s}} \right) = \frac{\zeta(s)}{\zeta(ks)}.$$ 

The series and product for $F$ are absolutely convergent in $\sigma > 1$ and recognizing that $F(s) = \zeta(s)/\zeta(ks)$ provides a meromorphic continuation of $F(s)$ to $\text{Re}(s) > 1/k$, and in this region there is only a simple pole at $s = 1$ with residue $1/\zeta(k)$.

Now we use Perron’s formula to obtain the desired asymptotic formula. Note that by the quantitative version of the formula, derived in class, we have that with $c = 1 + 1/\log x$, and assuming that $x$ is an integer + half,

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s)x^s \frac{ds}{s} = \sum_{n \leq x} \delta_k(n) + O\left( \sum_{n=1}^{\infty} \left( \frac{x}{n} \right) c \frac{1}{T|\log x/n|} \right).$$

Here $T \geq 2$ is a parameter that we’ll choose later; $T$ will be a power of $x$. As in class, the error term above may be estimated by dividing into the cases $x/2 < n < 2x$ (when $n$ is near $x$) and the complementary case (when $n$ is far away from $x$). For the second case we have that $|\log(x/n)| \gg 1$ and so the error here is

$$\ll \sum_n \left( \frac{x}{n} \right) c \frac{1}{T} \ll \frac{x}{T} \zeta(c) \ll \frac{x \log x}{T}.$$ 

For the first case we use that $|\log(x/n)| \gg |x - n|/x$ by a Taylor approximation. Hence, since also $1/2 \leq x/n \leq 2$ in this case, the error term is (using $x$ is an integer + half)

$$\ll \frac{1}{T} \sum_{x/2 \leq n \leq 2x} \frac{x}{|x - n|} \ll \frac{x \log x}{T}.$$ 

This completes our first step and we have that

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s)x^s \frac{ds}{s} = \sum_{n \leq x} \delta_k(n) + O\left( \frac{x \log x}{T} \right).$$
Now we move the line of integration to the line $\text{Re}(s) = \beta > 1/k$. Recall from class that we know for $\sigma > 0$ and $|s - 1| \geq 1/10$ that $|\zeta(\sigma + it)| \ll (1 + |t|)^{\max(0, 1 - \sigma)} \log(2 + |t|)$. Further in the region $\text{Re}(s) > 1/k$ we have that $1/\zeta(ks)$ is bounded, and so we conclude that $|F(s)| \ll (1 + |t|)^{\max(0, 1 - \sigma)} \log(2 + |t|)$ in this region. We now use this. When we move the line of integration to the line segment from $\beta - iT$ to $\beta + iT$ we pick up a main term which is the residue at $s = 1$ of $F(s) x/s$; that is a main term $x/\zeta(k)$. The error is the integral on the other three sides of the contour.

The horizontal lines contribute
\[
\ll \int_{\beta}^{c} x^{\sigma} T^{\max(0, 1 - \sigma)} \log T \frac{d\sigma}{T} \ll \frac{x \log T}{T} + x^{\beta - \beta} \log T.
\]

The integral on the $\beta$-line gives
\[
\ll \int_{-T}^{T} x^{\beta} T^{1 - \beta} \log T \frac{dt}{\beta + |t|} \ll x^{\beta} T^{1 - \beta} (\log T)^2.
\]

Putting everything together we obtain that the number of $k$-free integers up to $x$ equals
\[
\frac{x}{\zeta(2)} + O\left(\frac{x \log x}{T} + x^{\beta} T^{1 - \beta} (\log T)^2\right).
\]

Choose $T = x^{1 - \beta}$ and then the error is
\[
O(x^{1 - \beta} (\log x)^2).
\]

Choosing $\beta = 1/k + \epsilon$ for any $\epsilon > 0$ gives us the best error here, namely $O(x^{k/(2k - 1) + \epsilon})$.

2. We want to look at
\[
G(s) = \sum_{n=1}^{\infty} \left(\frac{n}{\phi(n)} \right)^{2010} \frac{1}{n^s},
\]
which by multiplicativity equals
\[
\prod_p \left(1 + \left(\frac{p}{p - 1}\right)^{2010} \frac{1}{p^s} \left(1 - \frac{1}{p^s}\right)^{-1}\right) = \zeta(s) \prod_p \left(1 + \frac{1}{p^s} \left(\left(\frac{p}{p - 1}\right)^{2010} - 1\right)\right).
\]

Call the Euler product above as $H(s)$. Since $(p/(p-1))^{2010} = 1 + O(1/p)$ we see that the Euler product defining $H(s)$ converges absolutely when $\text{Re}(s) > 0$. Moreover if $\sigma > 0$ then $|H(\sigma + it)|$ is bounded by $H(\sigma)$ which is just a constant (in terms of it’s dependence on $t$, and if we think of $\sigma$ as being uniformly bounded away from zero).

Now we can use Perron’s formula as in the last problem. Since the argument is very similar, we don’t give the details.
3. Since $\phi(n) \gg n^{1-\epsilon}$ the series defining $F(s)$ clearly converges absolutely for $\text{Re}(s) > 1$. Moreover by multiplicativity

$$F(s) = \prod_p \left(1 - \frac{1}{p^s} \right)^{-1} = \zeta(s) \prod_p \left(1 - \frac{1}{p^s} + \frac{1}{(p-1)^s}\right).$$

Let $G(s)$ denote the Euler product above. For any given $s$, if $p$ is sufficiently large (in terms of $|s|$) then $-1/p^s + 1/(p-1)^s = O(1/p^{\sigma+1})$. It follows that the Euler product for $G$ in fact converges absolutely for $\sigma > 0$. Thus we have that $F$ extends analytically to $\text{Re}(s) > 0$ except for a simple pole at $s = 1$ and its residue at 1 equals $G(1) = \prod_p (1 + 1/(p(p-1)))$. If we use the usual Perron’s formula argument we may expect that the number of $n$ with $\phi(n) \leq x$ is asymptotic to the residue at 1 of $x^s F(s)/s$ which is $xG(1)$. However there are difficulties in satisfactorily estimating $G(s)$ and the contour must be moved carefully. Such matters will be addressed by you in the Final!