SOLUTIONS FOR PROBLEM SET 7

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1. Put $\delta_k(n)$ to be 1 if n is k-free and 0 otherwise. Then $\delta_k(n)$ is a multiplicative function and

$$F(s) := \sum_{n=1}^{\infty} \frac{\delta_k(n)}{n^s} = \prod_p \Big(1 + \frac{1}{p^s} + \ldots + \frac{1}{p^{(k-1)s}}\Big) = \frac{\zeta(s)}{\zeta(ks)}.$$

The series and product for F are absolutely convergent in $\sigma > 1$ and recognizing that $F(s) = \zeta(s)/\zeta(ks)$ provides a meromorphic continuation of $F(s)$ to $\text{Re}(s) > 1/k$, and in this region there is only a simple pole at $s = 1$ with residue $1/\zeta(k)$.

Now we use Perron's formula to obtain the desired asymptotic formula. Note that by the quantitative version of the formula, derived in class, we have that with $c = 1 + 1/\log x$, and assuming that x is an integer + half,

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)x^s \frac{ds}{s} = \sum_{n\le x} \delta_k(n) + O\Big(\sum_{n=1}^{\infty}\Big(\frac{x}{n}\Big)^c \frac{1}{T|\log x/n|}\Big).$$

Here $T \ge 2$ is a parameter that we'll choose later; T will be a power of x. As in class, the error term above may be estimated by dividing into the cases $x/2 < n < 2x$ (when n is near x) and the complementary case (when n is far away from x). For the second case we have that $|\log(x/n)| \gg 1$ and so the error here is

$$\ll \sum_n \Big(\frac{x}{n}\Big)^c \frac{1}{T} \ll \frac{x}{T}\zeta(c) \ll \frac{x\log x}{T}.$$

For the first case we use that $|\log(x/n)| \gg |x-n|/x$ by a Taylor approximation. Hence, since also $1/2 \le x/n \le 2$ in this case, the error term is (using x is an integer + half)

$$\ll \frac{1}{T}\sum_{x/2\le n\le 2x} \frac{x}{|x-n|} \ll \frac{x\log x}{T}.$$

This completes our first step and we have that

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)x^s \frac{ds}{s} = \sum_{n\le x} \delta_k(n) + O\Big(\frac{x\log x}{T}\Big).$$

Now we move the line of integration to the line $\mathrm{Re}(s) = \beta > 1/k$. Recall from class that we know for $\sigma > 0$ and $|s-1| \geq 1/10$ that $|\zeta(\sigma+it)| \ll (1+|t|)^{\max(0,1-\sigma)} \log(2+|t|)$. Further in the region $\mathrm{Re}(s) > 1/k$ we have that $1/\zeta(ks)$ is bounded, and so we conclude that $|F(s)| \ll (1+|t|)^{\max(0,1-\sigma)} \log(2+|t|)$ in this region. We now use this. When we move the line of integration to the line segment from $\beta - iT$ to $\beta + iT$ we pick up a main term which is the residue at $s = 1$ of $F(s)x^s/s$; that is a main term $x/\zeta(k)$. The error is the integral on the other three sides of the contour.

The horizontal lines contribute

$$\ll \int_\beta^c x^\sigma \frac{T^{\max(0,1-\sigma)} \log T}{T} d\sigma \ll \frac{x \log T}{T} + x^\beta T^{-\beta} \log T.$$

The integral on the β-line gives

$$\ll \int_{-T}^{T} x^\beta T^{1-\beta} \log T \frac{dt}{\beta + |t|} \ll x^\beta T^{1-\beta} (\log T)^2.$$

Putting everything together we obtain that the number of k-free integers up to$x$ equals

$$\frac{x}{\zeta(2)} + O\Big(\frac{x \log x}{T} + x^\beta T^{1-\beta} (\log T)^2\Big).$$

Choose $T = x^{\frac{1-\beta}{2-\beta}}$ and then the error is

$$O(x^{\frac{1}{2-\beta}} (\log x)^2).$$

Choosing $\beta = 1/k + \epsilon$ for any $\epsilon > 0$ gives us the best error here, namely $O(x^{k/(2k-1)+\epsilon})$.

2. We want to look at

$$G(s) = \sum_{n=1}^\infty \Big(\frac{n}{\phi(n)}\Big)^{2010} \frac{1}{n^s},$$

which by multiplicativity equals

$$\prod_p \Big(1+\Big(\frac{p}{p-1}\Big)^{2010} \frac{1}{p^s}\Big(1-\frac{1}{p^s}\Big)^{-1}\Big) = \zeta(s) \prod_p \Big(1+\frac{1}{p^s}\Big(\Big(\frac{p}{p-1}\Big)^{2010} -1\Big)\Big).$$

Call the Euler product above as $H(s)$. Since $(p/(p-1))^{2010} = 1+O(1/p)$ we see that the Euler product defining $H(s)$ converges absolutely when $\mathrm{Re}(s) > 0$. Moreover if $\sigma > 0$ then $|H(\sigma + it)|$ is bounded by $H(\sigma)$ which is just a constant (in terms of it's dependence on t, and if we think of σ as being uniformly bounded away from zero).

Now we can use Perron's formula as in the last problem. Since the argument is very similar, we don't give the details.

3. Since $\phi(n) \gg n^{1-\epsilon}$ the series defining $F(s)$ clearly converges absolutely for $\mathrm{Re}(s) > 1$. Moreover by multiplicativity

$$F(s) = \prod_p \Big(1 + \frac{1}{(p-1)^s}\Big(1 - \frac{1}{p^s}\Big)^{-1}\Big) = \zeta(s) \prod_p \Big(1 - \frac{1}{p^s} + \frac{1}{(p-1)^s}\Big).$$

Let $G(s)$ denote the Euler product above. For any given s, if p is sufficiently large (in terms of $|s|$) then $-1/p^s + 1/(p-1)^s = O(1/p^{\sigma+1})$. It follows that the Euler product for G in fact converges absolutely for $\sigma > 0$. Thus we have that F extends analytically to $\mathrm{Re}(s) > 0$ except for a simple pole at $s = 1$ and its residue at 1 equals $G(1) = \prod_p (1 + 1/(p(p-1)))$. If we use the usual Perron's formula argument we may expect that the number of n with $\phi(n) \le x$ is asymptotic to the residue at 1 of $x^s F(s)/s$ which is $xG(1)$. However there are difficulties in satisfactorily estimating $G(s)$ and the contour must be moved carefully. Such matters will be addressed by you in the Final!