SOLUTIONS FOR PROBLEM SET 6

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1. Recall from class that for $\text{Re}(s) > 0$ we have

$$\zeta(s) = \sum_{n\le N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - s\int_{N+}^{\infty} \frac{\{y\}}{y^{s+1}} dy,$$

where N is a positive integer. Using this with $N = [|t|]$ (note that $|t| \ge 1$) we obtain that

$$\zeta(1+it) = \sum_{n\le [|t|]} \frac{1}{n^{1+it}} + O\Big(\frac{1}{|t|}\Big) + O\Big(|t| \int_{|t|}^{\infty} \frac{dy}{y^2}\Big) = \sum_{n\le |t|} \frac{1}{n^{1+it}} + O(1).$$

By the triangle inequality it follows that

$$|\zeta(1+it)| \le \sum_{n\le |t|} \frac{1}{n} + O(1) = \log |t| + O(1).$$

2. Differentiating our expression for $\zeta(s)$ above we find that (assuming I differentiated correctly)

$$\zeta'(s) = \sum_{n\le N} \frac{-\log n}{n^s} - \frac{N^{1-s}}{s-1}\Big(\log N + \frac{1}{(s-1)}\Big) + \int_{N+}^{\infty} \frac{\{y\}}{y^{s+1}}(-1+s\log y)dy.$$

Evaluating this for $s = 1+it$, and taking $N = [(1+|t|)\log(2+|t|)]$ say we find that

$$\zeta'(1+it) = -\sum_{n\le N} \frac{\log n}{n^{1+it}} + O(1) + O\Big(\int_N^{\infty} \frac{1+|t|\log y}{y^2} dy\Big) = -\sum_{n\le N} \frac{\log n}{n^{1+it}} + O(1).$$

Using the triangle inequality we again find that

$$|\zeta'(1+it)| \le \sum_{n\le N} \frac{\log n}{n} + O(1) \ll (\log N)^2.$$

3. Observe that, by the triangle inequality,

$$1 - \frac{1}{p^\sigma} \le \Big|1 - \frac{1}{p^{\sigma+it}}\Big| \le 1 + \frac{1}{p^\sigma}.$$

Therefore if $\sigma > 1$ multiplying these inequalities over all primes we obtain

$$\prod_p \Big(1+\frac{1}{p^\sigma}\Big)^{-1} \le |\zeta(\sigma+it)| \le \prod_p \Big(1-\frac{1}{p^\sigma}\Big)^{-1}.$$

Since $(1+1/p^\sigma)^{-1} = (1-1/p^\sigma)/(1-1/p^{2\sigma})$, the stated bounds follow.

4. To each $0 \le n \le N^K$ we associate the point $(\{\alpha_1 n\}, \{\alpha_2 n\}, \ldots, \{\alpha_K N\})$. In this manner there are $N+1$ points lying in N^K cuboids. By the pigeonhole principle two points must lie in the same cuboid. Say these correspond to the numbers m and n. We may see that for each $j \le K$, $\alpha_j(m-n)$ lies within $1/N$ of an integer. This gives the result.

5. Actually the problem is a little easier than the hint suggests, and you can just use the Dirichlet series rather than the Euler product; it is in some ways better to think of the Euler product though we won't go into this.

Since $\sigma > 1$ we may find N so large that for s with real part σ

$$\Big|\zeta(s) - \sum_{n\le N}\frac{1}{n^s}\Big| \le \frac{\epsilon}{3}.$$

We now show that there exists $T \ne 0$ such that

$$\Big|\sum_{n\le N}\Big(\frac{1}{n^{\sigma+it}} - \frac{1}{n^{\sigma+it+iT}}\Big)\Big| \le \frac{\epsilon}{3}, \tag{1}$$

for all $t \in \mathbb{R}$, and then our claimed result would follow.

Suppose T is such that $|n^{iT}-1| \le \epsilon/(3\zeta(\sigma))$ for every $1 \le n \le N$. Then we would have

$$\Big|\sum_{n\le N}\Big(\frac{1}{n^{\sigma+it}} - \frac{1}{n^{\sigma+it+iT}}\Big)\Big| \le \sum_{n\le N}\frac{1}{n^\sigma}|n^{iT}-1| \le \frac{\epsilon}{3},$$

as desired.

Why does T with the above property exist? Here we use problem 4. To make the connection, note that $|e^{2\pi i\theta}-1| = 2|\sin(\pi\theta)| \le 2\pi\|\theta\|$, since $|\sin\pi\theta| = |\sin(\pi\|\theta\|)|$, and $|\sin x| \le |x|$. Therefore $|n^{iT}-1| = |e^{2\pi i(T\log n/2\pi)}-1| \le 2\pi\|(T\log n)/2\pi\|$. So we only need to find T such that $\|T\frac{\log n}{2\pi}\| \le \epsilon/(6\pi\zeta(\sigma))$ for all $n \le N$ and the result would follow. The existence of such non-zero T is now guaranteed by problem 4.