SOLUTIONS FOR PROBLEM SET 5

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1. The Dirichlet series for $Q(s)$ converges absolutely in $\sigma > 1$. Formally the Euler product for $Q(s)$ is $\prod_p (1+p^{-s})$ and we see that this converges absolutely in $\sigma > 1$. Since $(1+p^{-s}) = (1-p^{-2s})/(1-p^{-s})$, in the region of absolute convergence, we see that $Q(s) = \zeta(s)/\zeta(2s)$. Now we know that $(s-1)\zeta(s)$ is analytic in $\sigma > 0$ (and in fact for all s), and also $\zeta(2s)$ is analytic in $\sigma > 1/2$ and is free of zeros there. Therefore $(s-1)\zeta(s)/\zeta(2s)$ is analytic in $\sigma > 1/2$ and we have given an analytic continuation of $(s-1)Q(s)$ to this region. Potentially $(s-1)Q(s)$ is analytic in the wider region $\sigma > 1/4$ but this relies upon knowing that $\zeta(2s)$ is not zero in that region which is the content of the Riemann hypothesis!

2. This should be a clear application of Euler-Maclaurin, and in any case we went over a similar argument in class; so I'll skip this solution.

3. Since $\log(1-z) = -z + O(z^2)$ for $|z|$ small, we see that

$$\sum_{p>x} \log\Big(1 - \frac{1}{p^\sigma}\Big)^{-1} = \sum_{p>x}\Big(\frac{1}{p^\sigma} + O\Big(\frac{1}{p^{2\sigma}}\Big)\Big) = \sum_{p>x}\frac{1}{p^\sigma} + O\Big(\frac{1}{x}\Big).$$

To evaluate the sum above we use partial summation using the knowledge that $\sum_{p\le z} (\log p)/p = \log z + O(1)$. Thus the sum we want equals

$$\begin{aligned}\int_x^\infty \frac{1}{t^{\sigma-1}\log t} d\Big(\sum_{p\le t}\frac{\log p}{p}\Big) &= \Big[\sum_{p\le t}\frac{\log p}{p}\frac{1}{t^{\sigma-1}\log t}\Big]_x^\infty \\ &\quad -\int_x^\infty \sum_{p\le t}\frac{\log p}{p}\Big(-\frac{1}{t^\sigma(\log t)^2} - \frac{\sigma-1}{t^\sigma \log t}\Big)dt.\end{aligned}$$

Making use of the asymptotic formula for $\sum_{p\le t}(\log p)/p$ we see that the above equals

$$-\frac{(\log x + O(1))}{x^{\sigma-1}\log x} + \int_x^\infty (\log t + O(1))\Big(\frac{1}{t^\sigma(\log t)^2} + \frac{\sigma-1}{t^\sigma\log t}\Big)dt.$$

Now we clean this up a bit: the error terms give in all $O(1/\log x)$ and the main term, recalling $\sigma = 1 + 1/\log x$, equals

$$-\frac{1}{e} + \int_1^\infty \Big(\frac{1}{t^\sigma \log t} + \frac{\sigma - 1}{t^\sigma}\Big) dt = \int_x^\infty \frac{1}{t^\sigma \log t} dt.$$

Making the change of variables $\log t = y \log x$, we obtain that the above equals $\int_1^\infty (e^{-y}/y) dy$, as needed.

4. This is similar to problem 3, but slightly more complicated. First by using the Taylor expansion $\log(1-t)^{-1} = \sum_{k=1}^\infty t^k/k$ we see that

$$\log\Big(1 - \frac{1}{p^\sigma}\Big)^{-1} - \log\Big(1 - \frac{1}{p}\Big)^{-1} = -\sum_{k=1}^\infty \frac{1}{k}\Big(\frac{1}{p^k} - \frac{1}{p^{k\sigma}}\Big).$$

Now note that for all $t > 0$ we have $1 - e^{-t} = O(t)$ (check), and so

$$\frac{1}{p^k} - \frac{1}{p^{k\sigma}} = \frac{1}{p^k}\Big(1 - \frac{1}{p^{k(\sigma-1)}}\Big) = O\Big(\frac{k(\sigma-1)\log p}{p^k}\Big).$$

Using this for $k \geq 2$ we find that

$$\begin{aligned}\log\Big(1 - \frac{1}{p^\sigma}\Big)^{-1} - \log\Big(1 - \frac{1}{p}\Big)^{-1} &= -\Big(\frac{1}{p} - \frac{1}{p^\sigma}\Big) + O\Big(\sum_{k\geq 2} \frac{(\sigma-1)\log p}{p^k}\Big) \\ &= -\Big(\frac{1}{p} - \frac{1}{p^\sigma}\Big) + O\Big(\frac{\log p}{p^2 \log x}\Big).\end{aligned}$$

Thus we obtain that

$$\sum_{p\leq x} \log\Big(1 - \frac{1}{p^\sigma}\Big)^{-1} - \log\Big(1 - \frac{1}{p}\Big)^{-1} = -\sum_{p\leq x}\Big(\frac{1}{p} - \frac{1}{p^\sigma}\Big) + O\Big(\frac{1}{\log x}\Big).$$

Now follow the argument of problem 3, using partial summation together with information on $\sum_{p\leq z}(\log p)/p$. I'll skip these details!

5. If we add the results of problems 4 and 5 we obtain that

$$\log\zeta(\sigma) - \sum_{p\leq x} \log\Big(1 - \frac{1}{p}\Big)^{-1} = -\int_0^1 \frac{1 - e^{-t}}{t} dt + \int_1^\infty \frac{e^{-t}}{t} dt + O\Big(\frac{1}{\log x}\Big).$$

Since $\zeta(\sigma) = \frac{1}{\sigma - 1} + \gamma + \ldots = \log x + O(1)$, we obtain the stated result.