1. If $\omega(q) = k$ then we claim that

$$\frac{\phi(q)}{q} \geq \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right),$$

where $p_j$ denotes the $j$-th smallest prime ($p_1 = 2$, $p_2 = 3$ etc). The inequality above holds because $\phi(q)/q = \prod_{p \mid q} (1 - 1/p)$ and there are, by assumption, $k$ primes involved in the product, and this product is clearly smallest when the primes involved are the first $k$ primes.

By Mertens’s theorem we conclude that

$$\frac{\phi(q)}{q} \geq e^{-\gamma} + o(1) \log p_k.$$ 

Now if $\omega(q) = k$ then certainly $q \geq \prod_{j=1}^{k} p_j = \exp(\vartheta(p_k)) \geq 2^{p_k(1+o(1))}$ by our Chebyshev bound. Thus $p_k \leq (1 + o(1)) \log q/\log 2$ and substituting this in our lower bound for $\phi(q)/q$ we conclude that

$$\frac{\phi(q)}{q} \geq e^{-\gamma} + o(1) \log p_k \geq e^{-\gamma} + o(1) \log \log q.$$ 

2. The 3-divisor function is multiplicative. Let us now determine its values on the prime powers:

$$d_3(p^k) = \sum_{a=0}^{k} d_2(p^a) = \sum_{a=0}^{k} (a + 1) = \binom{k+2}{2}.$$ 

Therefore

$$\frac{d_3(n)}{n^\epsilon} = \prod_{p^b \mid n} \left(\frac{k+2}{2}\right) p^{-kb}.$$ 

For any fixed $\epsilon > 0$, and a prime $p$ consider the sequence of numbers $\binom{k+2}{2} p^{-kb}$ for $k = 0, 1, 2, \ldots$. We may check easily that the sequence increases initially to a maximum, and then decreases to zero. Also if $p$ is sufficiently large the maximum is attained at $k = 0$ and the sequence is monotone decreasing. If we set $C(p, \epsilon) = \max_{k \geq 0} \binom{k+2}{2} p^{-kb}$ then it
follows that
\[ \frac{d_3(n)}{n^\epsilon} \leq \prod_p C(p, \epsilon) =: C(\epsilon), \]
where the product makes sense because \( C(p, \epsilon) = 1 \) for all but finitely many primes.

If \( p > 79 \) (so \( p \geq 83 \)) we have that \( d_3(p) / p^{1/4} < 1 \) and for such \( p \) we have \( C(p, 1/4) = 1 \). If \( 17 \leq p \leq 79 \) then we have \( d_3(p) / p^{1/4} > d_3(p^2) / p^{2/4} \) and so for such \( p \)
\[ C(p, 1/4) = \frac{3}{p^{1/4}}. \]

If \( p = 11 \) or \( 13 \) we may check that \( C(p, 1/4) = 6/p^{1/2} \). Next by explicit computer calculations I found that \( C(7, 1/4) = 10/7^{3/4}, C(5, 1/4) = 15/5 = 3, C(3, 1/4) = 28/3^{1/4} \) and \( C(2, 1/4) = 66/2^{5/2}. \) Thus a permissible value for \( C \) is
\[ \prod_{p \leq 79} C(p, 1/4) = 17653.12867 \ldots . \]

Note that this value is tight for \( n = 2^{10} \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 79. \)

3. The polynomial \( P_1(x) = (x^2(1 - x)^2(2x - 1))^2 \) is non-negative on \([0, 1]\) and equals zero only at 0, 1, and 1/2. A little calculus shows that on \([0, 1]\) this polynomial attains its maxima at \((5 \pm \sqrt{5})/10\) and this maximum value equals \( 5^{-5}. \) Therefore \( P_N(x) = (x^2(1 - x)^2(2x - 1))^{2N} \) is a non-negative polynomial of degree \( 10N \) and its maximum value on \([0, 1]\) equals \( 5^{-5N}. \) Hence
\[ 0 < \int_0^1 P_N(x) dx < 5^{-5N}. \]

We may expand \( P_N(x) = \sum_{j=0}^{10N} a_j x^j, \) and note that the coefficients \( a_j \) are integers. Therefore
\[ \int_0^1 P_N(x) = \sum_{j=0}^{10N} \frac{a_j}{j + 1} \]
is a rational number \( a/b \) say (in lowest terms), and the denominator \( b \) is a divisor of the least common multiple of the numbers from 1 to \( 10N + 1. \) Thus the denominator \( b \) is at most \( \exp(\psi(10N + 1)). \)

We deduce from the above observations that
\[ 1 \leq a = b \int_0^1 P_N(x) dx < \exp(\psi(10N + 1))5^{-5N}, \]
and therefore
\[ \psi(10N + 1) > (5 \log 5)N. \]
Since this holds for every integer \( N \) we conclude that
\[ \psi(x) \geq \frac{\log 5}{2} x + O(1), \]
and so \( a \geq \log \sqrt{5} = 0.8047 \ldots \).