SOLUTIONS FOR PROBLEM SET 4

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1. If $\omega(q) = k$ then we claim that

$$\frac{\phi(q)}{q} \ge \prod_{j=1}^{k} \Big(1 - \frac{1}{p_j}\Big),$$

where p_j denotes the j-th smallest prime ($p_1 = 2$, $p_2 = 3$ etc). The inequality above holds because $\phi(q)/q = \prod_{p|q}(1 - 1/p)$ and there are, by assumption, k primes involved in the product, and this product is clearly smallest when the primes involved are the first k primes.

By Mertens's theorem we conclude that

$$\frac{\phi(q)}{q} \ge \frac{e^{-\gamma} + o(1)}{\log p_k}.$$

Now if $\omega(q) = k$ then certainly $q \ge \prod_{j=1}^{k} p_j = \exp(\vartheta(p_k)) \ge 2^{p_k(1+o(1))}$ by our Chebyshev bound. Thus $p_k \le (1 + o(1)) \log q/ \log 2$ and substituting this in our lower bound for $\phi(q)/q$ we conclude that

$$\frac{\phi(q)}{q} \ge \frac{e^{-\gamma} + o(1)}{\log p_k} \ge \frac{e^{-\gamma} + o(1)}{\log \log q}.$$

2. The 3-divisor function is multiplicative. Let us now determine its values on the prime powers:

$$d_3(p^k) = \sum_{a=0}^{k} d_2(p^a) = \sum_{a=0}^{k} (a + 1) = \binom{k+2}{2}.$$

Therefore

$$\frac{d_3(n)}{n^\epsilon} = \prod_{p^k \| n} \binom{k+2}{2} p^{-k\epsilon}.$$

For any fixed $\epsilon > 0$, and a prime p consider the sequence of numbers $\binom{k+2}{2}p^{-k\epsilon}$ for $k = 0, 1, 2, \ldots$. We may check easily that the sequence increases initially to a maximum, and then decreases to zero. Also if p is sufficiently large the maximum is attained at $k = 0$ and the sequence is monotone decreasing. If we set $C(p, \epsilon) = \max_{k\ge0} \binom{k+2}{2}p^{-k\epsilon}$ then it follows that

$$\frac{d_3(n)}{n^\epsilon} \le \prod_p C(p,\epsilon) =: C(\epsilon),$$

where the product makes sense because $C(p,\epsilon) = 1$ for all but finitely many primes.

If $p > 79$ (so $p \ge 83$) we have that $d_3(p)/p^{1/4} < 1$ and for such p we have $C(p, 1/4) = 1$. If $17 \le p \le 79$ then we have $d_3(p)/p^{1/4} > d_3(p^2)/p^{2/4}$ and so for such p

$$C(p, 1/4) = \frac{3}{p^{1/4}}.$$

If $p = 11$ or 13 we may check that $C(p, 1/4) = 6/p^{1/2}$. Next by explicit computer calculations I found that $C(7, 1/4) = 10/7^{3/4}$, $C(5, 1/4) = 15/5 = 3$, $C(3, 1/4) = 28/3^{3/2}$ and $C(2, 1/4) = 66/2^{5/2}$. Thus a permissible value for C is

$$\prod_{p\le 79} C(p, 1/4) = 17653.12867\ldots.$$

Note that this value is tight for $n = 2^{10} \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17 \cdots 79$.

3. The polynomial $P_1(x) = (x^2(1-x)^2(2x-1))^2$ is non-negative on $[0,1]$ and equals zero only at 0, 1, and 1/2. A little calculus shows that on $[0,1]$ this polynomial attains its maxima at $(5 \pm \sqrt{5})/10$ and this maximum value equals 5^{-5}. Therefore $P_N(x) = (x^2(1-x)^2(2x-1))^{2N}$ is a non-negative polynomial of degree $10N$ and its maximum value on $[0,1]$ equals 5^{-5N}. Hence

$$0 < \int_0^1 P_N(x)dx < 5^{-5N}.$$

We may expand $P_N(x) = \sum_{j=0}^{10N} a_j x^j$, and note that the coefficients a_j are integers. Therefore

$$\int_0^1 P_N(x) = \sum_{j=0}^{10N} \frac{a_j}{j+1}$$

is a rational number a/b say (in lowest terms), and the denominator b is a divisor of the least common multiple of the numbers from 1 to $10N + 1$. Thus the denominator b is at most $\exp(\psi(10N + 1))$.

We deduce from the above observations that

$$1 \le a = b\int_0^1 P_N(x)dx < \exp(\psi(10N + 1))5^{-5N},$$

and therefore

$$\psi(10N + 1) > (5 \log 5)N.$$

Since this holds for every integer N we conclude that

$$\psi(x) \geq \frac{\log 5}{2} x + O(1),$$

and so $a \geq \log \sqrt{5} = 0.8047\ldots$.