SOLUTIONS FOR PROBLEM SET 3

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1. Suppose first that n is square-free. Then the RHS is simply

$$-\sum_{ap=n} \mu(a) \log p = \mu(n) \sum_{p|n} \log p = \mu(n) \log n,$$

as desired.

Suppose now that n is divisible by the square of two different primes. Then the LHS is zero, and so is the RHS because if we write $n = ab$ then $\Lambda(b) = 0$ unless b is a prime power, and in that case $\mu(a) = 0$ because a will be divisible by the square of a prime.

Lastly suppose that $n = p^k m$ where $k \geq 2$ and m is square-free with $p \nmid m$. Here the LHS is zero, and the RHS equals

$$-\mu(m)\Lambda(p^k) - \mu(pm)\Lambda(p^{k-1}) = -\mu(m)\log p + \mu(m)\log p = 0.$$

Having exhausted all cases, we are done.

2. Let m and n be coprime integers, and let h denote $f * g$. We wish to show that h is multiplicative; that is, $h(mn) = h(m)h(n)$. Note that if d is a divisor of mn then d may be written uniquely as $k\ell$ where $k|m$ and $\ell|n$. Conversely given $k|m$ and $\ell|n$ there corresponds the unique divisor $k\ell$ of mn. Thus

$$h(mn) \;\; = \;\; \sum_{d|mn} f(d)g(mn/d) = \sum_{k|m} \sum_{\ell|n} f(k\ell) g\Big(\frac{m}{k}\frac{n}{\ell}\Big).$$

Since f and g are multiplicative, and $(k,\ell) = 1$ and $(m/k, n/\ell) = 1$ we see that the above equals

$$\sum_{k|m} f(k)g(m/k) \sum_{\ell|n} f(\ell)g(n/\ell) = h(m)h(n).$$

Thus the convolution of two multiplicative functions is multiplicative.

The convolution of two completely multiplicative functions need not be completely multiplicative. Let 1 denote the function $1(n) = 1$ for all integers n. Then the function 1 is completely multiplicative, but $(1 * 1)(n)$ is the divisor function $d(n)$ which is not completely multiplicative.

3. As hinted in the problem let us try and use multiplicativity. The LHS of the stated identity is a multiplicative function. The RHS is the

convolution of two multiplicative functions: the divisor function, and the function $g(d)$ which is 0 unless d is a square, and equals $\mu(k)$ if $d = k^2$. By problem 2 the convolution of g with the divisor function is multiplicative, and so the RHS is multiplicative also. Therefore it suffices to check this identity when n is the power of a prime; say $n = p^k$ with $k \geq 1$. In this case the LHS equals 2. Let us compute the RHS. If $k = 1$ then we easily see that it equals 2, while if $k \geq 2$ it equals

$$d(p^k) + \mu(p)d(p^{k-2}) = (k+1) - (k-2+1) = 2$$

also. This proves our identity.

From our identity we obtain that

$$\sum_{n\leq x} 2^{\omega(n)} = \sum_{n\leq x} \sum_{d^2 m = n} \mu(d)d(m) = \sum_{d^2\leq x} \mu(d) \sum_{m\leq x/d^2} d(m).$$

In the inner sum over m we use the asymptotic formula established in class:

$$\sum_{m\leq z} d(m) = z\log z + (2\gamma - 1)z + O(\sqrt{z}).$$

Thus we find that

$$\sum_{n\leq x} 2^{\omega(n)} = \sum_{d\leq\sqrt{x}} \mu(d)\Big(\frac{x}{d^2}\log\frac{x}{d^2} + (2\gamma-1)\frac{x}{d^2} + O\Big(\frac{\sqrt{x}}{d}\Big)\Big).$$

First we dispense with the error term above: it is

$$O\Big(\sqrt{x}\sum_{d\leq x}\frac{1}{d}\Big) = O(\sqrt{x}\log x),$$

as desired. As for the main term this may be written as

$$(x\log x + (2\gamma-1)x)\sum_{d\leq\sqrt{x}}\frac{\mu(d)}{d^2} - x\sum_{d\leq\sqrt{x}}\frac{\mu(d)}{d^2}\log d^2.$$

In the first term above we extend the sum to infinity, and so the first term gives

$$(x\log x+(2\gamma-1)x)\Big(\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}+O\Big(\frac{1}{\sqrt{x}}\Big)\Big) = \frac{6}{\pi^2}(x\log x+(2\gamma-1)x)+O(\sqrt{x}\log x).$$

We also extend the sum in the second term to infinity and we get

$$-x\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}(2\log d) + O\Big(x\sum_{d>\sqrt{x}}\frac{\log d}{d^2}\Big) = Cx + O(\sqrt{x}\log x),$$

for some constant $C = -2\sum_d (\log d)\mu(d)/d^2$. Putting everything together we obtain the stated result.

4. (a) As noted in the problem either $n \le m$ or $m \le n$, and when $m = n = 1$ both possibilities occur. Therefore, by symmetry, our sum is

$$-1 + 2\sum_{n\le x}\sum_{\substack{m\le n\\(m,n)=1}} 1 = -1 + 2\sum_{n\le x}\phi(n).$$

(b). Note that $\phi(n) = \sum_{d|n}\mu(d)n/d = \sum_{dm=n}\mu(d)m$. Therefore

$$\sum_{n\le x}\phi(n) = \sum_{d\le x}\mu(d)\sum_{m\le x/d} m.$$

The inner sum over m is easily seen to be $(x/d)^2/2+O(x/d)$. Therefore our sum is

$$\begin{aligned}\frac{x^2}{2}\sum_{d\le x}\frac{\mu(d)}{d^2} + O\Big(x\sum_{d\le x}\frac{1}{d}\Big) &= \frac{x^2}{2}\Big(\sum_{d=1}^{\infty}\frac{\mu(d)}{d^2} + O\Big(\frac{1}{x}\Big)\Big) + O(x\log x)\\ &= \frac{3}{\pi^2}x + O(x\log x).\end{aligned}$$

(c). This is just a matter of interpretation: If m and n are chosen randomly from the interval $[1, x]$ then the probability that they are coprime is exactly theLHS of problem (a) divided by x^2, and from (a) and (b) this is $= 6/\pi^2 + O((\log x)/x)$.

5. Note that $n/\phi(n)$ is a multiplicative function, and so is $\mu(d)^2/\phi(d)$. Thus by problem 2 both sides of the identity to be proved are multiplicative. Therefore we need only check that they match for prime powers p^k with $k \ge 1$. Note that in this case the LHS equals

$$\frac{p^k}{\phi(p^k)} = \frac{p}{p-1}.$$

The RHS equals

$$1 + \frac{1}{\phi(p)} = 1 + \frac{1}{p-1} = \frac{p}{p-1}.$$

This verifies our identity.

Using our identity and interchanging summations we obtain that

$$\sum_{n\le x}\frac{n}{\phi(n)} = \sum_{n\le x}\sum_{d|n}\frac{\mu(d)^2}{\phi(d)} = \sum_{d\le x}\frac{\mu(d)^2}{\phi(d)}\Big[\frac{x}{d}\Big] = \sum_{d\le x}\frac{\mu(d)^2}{\phi(d)}\Big(\frac{x}{d} + O(1)\Big).$$

First let us handle the error term above which is $O(\sum_{d\le x}\mu(d)^2/\phi(d))$. Now we suspect that $\phi(d)$ is usually about size d (after all the problem is to show that it's average value is c, and in any case in class we showed that $\phi(n)/n \ge (e^{-\gamma} + o(1))/\log\log n)$ and this explains why we may expect the error term to be $O(\log x)$. To prove this rigorously note that

$$\sum_{d \le x} \frac{\mu(d)^2}{\phi(d)} \le \prod_{p \le x} \Big(1 + \frac{\mu(p)^2}{\phi(p)}\Big),$$

because when we expand the RHS out we get the sum of $\mu(n)^2/\phi(n)$ for all n that are composed only of primes below x, and certainly every $n \le x$ is included in this. Therefore the error term is

$$O\Big(\prod_{p \le x}\Big(1 + \frac{1}{p-1}\Big)\Big) = O\Big(\prod_{p \le x}\Big(1 - \frac{1}{p}\Big)^{-1}\Big) = O(\log x),$$

by Mertens's theorem. If you didn't find this trick but got an error term of $O(\log x \log\log x)$ using our result from class, that's good enough for me!

Now look at the main term which is $x\sum_{d \le x} \mu(d)^2/(d\phi(d))$. We extend the sum over d to all integers, and define c to be the value of $\sum_{d=1}^{\infty} \mu(d)^2/(d\phi(d))$. Note that the series converges because $\phi(d) \gg d/\log\log d$ from class. The error involved in extending the sum is

$$O\Big(x\sum_{d > x} \frac{\mu(d)^2}{d\phi(d)}\Big) = O\Big(x\sum_{d > x} \frac{\log\log d}{d^2}\Big) = O(\log\log x),$$

by using our lower bound for $\phi(n)$, and I'll leave you to justify the final step. We could be more careful and in fact $\sum_{d > x} \frac{\mu(d)^2}{d\phi(d)} = O(1/x)$ but it doesn't matter.

We have now established that

$$\sum_{n \le x} \frac{n}{\phi(n)} = cx + O(\log x),$$

with

$$c = \sum_{d=1}^{\infty} \frac{\mu(d)^2}{d\phi(d)}.$$

It remains lastly to verify that this value of c equals $\zeta(2)\zeta(3)/\zeta(6)$. To see this note that $\mu(d)^2/(d\phi(d))$ is multiplicative and therefore we may write

$$\sum_{d=1}^{\infty} \frac{\mu(d)^2}{d\phi(d)} = \prod_p \Big(1 + \frac{\mu(p)^2}{p\phi(p)}\Big) = \prod_p \Big(1 + \frac{1}{p(p-1)}\Big).$$

On the other hand

$$\begin{aligned}\frac{\zeta(2)\zeta(3)}{\zeta(6)} &= \prod_p \Big(1-\frac{1}{p^2}\Big)^{-1}\Big(1-\frac{1}{p^3}\Big)^{-1}\Big(1-\frac{1}{p^6}\Big) = \prod_p \frac{(1+1/p^3)}{(1-1/p^2)} \\ &= \prod_p \frac{p^3+1}{p(p^2-1)} = \prod_p \frac{p^2-p+1}{p(p-1)} = \prod_p \Big(1+\frac{1}{p(p-1)}\Big).\end{aligned}$$

The fact that the algebra works out nicely is simply a matter of good fortune, and doesn't mean anything!