MATH 155: PROBLEM SET 8

Due March 4

1. In class I discussed why

$$\lim_{\sigma^{1+}} \sum_{n=1}^{\infty} \frac{\mu(n)}{n}(1-n^{1-\sigma}) = 0.$$

Provide a careful proof.

2. Using induction, or otherwise, prove that

$$\pi_k(x) \sim \frac{x}{\log x} \frac{(\log\log x)^{k-1}}{(k-1)!}$$

where $\pi_k(x)$ counts the number of integers n below x of the form $n = p_1 \cdots p_k$.

3. Let $\chi_{-4}(n) = 1$ if $n \equiv 1 \pmod 4$, -1 if $n \equiv 3 \pmod 4$ and 0 if n is even. The function χ_{-4} is an example of a Dirichlet character. Let $L(s, \chi_{-4}) = \sum_{n=1}^{\infty} \chi_{-4}(n)/n^s = \prod_p (1-\chi_{-4}(p)/p^s)^{-1}$. Prove that $L(s,\chi_{-4})$ extends analytically to $\mathrm{Re}(s) > 0$, and work out bounds for $L(s,\chi_{-4})$ in this region (analogous to bounds for $\zeta(s)$).

4. Let $L(s,\chi_{-4})$ be as in problem 3. Prove that $L(1+it,\chi_{-4}) \neq 0$ for all $t \in \mathbb{R}$. You may need to consider separately the case when $t = 0$. Prove also a lower bound for $L(\sigma+it,\chi_{-4})$ when $\sigma > 1$ (analogous to our lower bound for $\zeta(s)$).

5. Prove that

$$\sum_{n\le x} \mu(n)\chi_{-4}(n) \ll \frac{x}{(\log x)^A}$$

for any $A > 0$. Deduce that

$$\sum_{\substack{n\le x\\ n\equiv 1 \pmod 4}} \mu(n) \ll \frac{x}{(\log x)^A}.$$

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