MATH 155: PROBLEM SET 7

DUE FEBRUARY 25

1. Using the method of Perron's formula and contour integration derive an asymptotic formula for the number of k-free integers up to x. Recall that a number is k-free if it is not divisible by the k-th power of any prime. Here $k \geq 2$ is an integer.

2. Prove that there is a constant $c > 0$ such that

$$\sum_{n \leq x} \left(\frac{n}{\phi(n)} \right)^{2010} = cx + O(x^{1-\delta})$$

for some $\delta > 0$.

3. Consider the series

$$F(s) = \sum_{n=1}^{\infty} \frac{1}{\phi(n)^s}.$$

Where does this converge absolutely? Show that $F(s)$ extends meromorphically to the region $\mathrm{Re}(s) > 0$, and is analytic except a simple pole at $s = 1$. Make a guess about how many integers n are there with $\phi(n) \leq x$. What issues would you face if you want to make a rigorous proof of your guess?