1. Let \(|t| \geq 1\). Prove that
\[
\zeta(1 + it) = \sum_{n \leq |t|} \frac{1}{n^{1+it}} + O(1).
\]
Conclude that
\[
|\zeta(1 + it)| \leq \log |t| + O(1).
\]

2. For \(|t| \geq 1\) obtain, as in problem 1, an approximation for \(\zeta'(1 + it)\), and deduce an estimate for \(|\zeta'(1 + it)|\).

3. Prove, using the Euler product or otherwise, that for \(\sigma > 1\)
\[
\zeta(\sigma) \geq |\zeta(\sigma + it)| \geq \frac{\zeta(2\sigma)}{\zeta(\sigma)}.
\]

4. Let \(\|x\| := \min_{n \in \mathbb{Z}} |x - n|\) denote the distance between \(x\) and the nearest integer. Suppose you are given real numbers \(\alpha_1, \ldots, \alpha_K\). For any integer \(N \geq 1\) prove that there exists \(n\) with \(1 \leq n \leq N^K\) such that \(\|n\alpha_j\| \leq 1/N\) for each \(j = 1, \ldots, K\). Hint: Divide the \(K\)-dimensional hypercube \([0, 1)^K\) into cuboids with side-length \(1/N\). For each \(0 \leq n \leq N^K\) associate a point in this hypercube; use the pigeonhole principle.

5. Let \(\sigma > 1\) be fixed, and suppose \(\epsilon > 0\) is given. Show that there exists a non-zero real number \(T = T(\sigma, \epsilon)\) such that
\[
|\zeta(\sigma + it) - \zeta(\sigma + it + iT)| \leq \epsilon
\]
for all real numbers \(t\). In other words, the zeta-function on the line \(\text{Re}(s) = \sigma\) is almost periodic, and \(T\) is called an \(\epsilon\)-almost period. Hint: First show that \(\zeta(\sigma + it)\) is well approximated by a suitable truncation of the Euler product, and then use the result of problem 4.