MATH 155: PROBLEM SET 6

Due February 18

1. Let $|t| \geq 1$. Prove that

$$\zeta(1+it) = \sum_{n \leq |t|} \frac{1}{n^{1+it}} + O(1).$$

Conclude that

$$|\zeta(1+it)| \leq \log |t| + O(1).$$

2. For $|t| \geq 1$ obtain, as in problem 1, an approximation for $\zeta'(1+it)$, and deduce an estimate for $|\zeta'(1+it)|$.

3. Prove, using the Euler product or otherwise, that for $\sigma > 1$

$$\zeta(\sigma) \geq |\zeta(\sigma+it)| \geq \frac{\zeta(2\sigma)}{\zeta(\sigma)}.$$

4. Let $\|x\| := \min_{n\in\mathbb{Z}} |x-n|$ denote the distance between x and the nearest integer. Suppose you are given real numbers $\alpha_1, \ldots, \alpha_K$. For any integer $N \geq 1$ prove that there exists n with $1 \leq n \leq N^K$ such that $\|n\alpha_j\| \leq 1/N$ for each $j = 1, \ldots, K$. Hint: Divide the K-dimensional hypercube $[0,1)^K$ into cuboids with side-length $1/N$. For each $0 \leq n \leq N^K$ associate a point in this hypercube; use the pigeonhole principle.

5. Let $\sigma > 1$ be fixed, and suppose $\epsilon > 0$ is given. Show that there exists a non-zero real number $T = T(\sigma, \epsilon)$ such that

$$|\zeta(\sigma+it) - \zeta(\sigma+it+iT)| \leq \epsilon$$

for all real numbers t. In other words, the zeta-function on the line $\mathrm{Re}(s) = \sigma$ is *almost periodic*, and T is called an ϵ-*almost period*. Hint: First show that $\zeta(\sigma+it)$ is well approximated by a suitable truncation of the Euler product, and then use the result of problem 4.