MATH 155: PROBLEM SET 5

Due February 11

1. Let $Q(s) = \sum_{n=1}^{\infty} \mu(n)^2/n^s$. In what region does Q converge absolutely? Does Q have an Euler product; in what region does that converge absolutely? Can you express Q in terms of the Riemann zeta-function? Analytically continue $(s-1)Q(s)$ to as large a region as you can.

2. Using Euler-Mclaurin or otherwise prove that for $\sigma > 1$ we have

$$\zeta(s) = \frac{s}{s-1} - \frac{1}{2} + \frac{s}{12} + s(s+1)\int_{1^-}^{\infty} \frac{(\{y\}^2/2 - \{y\}/2 + 1/12)}{y^{s+2}} dy.$$

Use the RHS to obtain a definition for $\zeta(s)$ in $\sigma > -1$ (and $s \neq 1$). What is $\zeta(0)$? Carry out the Euler-Mclaurin process one more time to obtain a definition of $\zeta(s)$ for $\sigma > -2$ (and $s \neq 1$). Determine $\zeta(-1)$.

3. In this problem put $\sigma = 1 + 1/\log x$. Show that

$$\sum_{p>x} \log\Big(1 - \frac{1}{p^\sigma}\Big)^{-1} = \sum_{p>x} \frac{1}{p^\sigma} + O\Big(\frac{1}{x}\Big) = \int_1^{\infty} \frac{e^{-t}}{t} dt + O\Big(\frac{1}{\log x}\Big).$$

Hint: use that $\sum_{p\le z} (\log p)/p = \log z + O(1)$.

4. Again put $\sigma = 1 + 1/\log x$. Show that

$$\sum_{p\le x} \Big(\log\Big(1 - \frac{1}{p^\sigma}\Big)^{-1} - \log\Big(1 - \frac{1}{p}\Big)^{-1}\Big) = -\int_0^1 \frac{1 - e^{-t}}{t} dt + O\Big(\frac{1}{\log x}\Big).$$

This may be a little challenging; one hint is to use the Taylor series for $\log(1-t)^{-1}$.

5. Using problems 4 and 5 and information about $\log \zeta(\sigma)$ from class, establish that

$$\sum_{p\le x} \log\Big(1 - \frac{1}{p}\Big)^{-1} = \log\log x + \int_0^1 \frac{1 - e^{-t}}{t} dt - \int_1^{\infty} \frac{e^{-t}}{t} dt + O\Big(\frac{1}{\log x}\Big).$$

Use a computer to evaluate the constant in the RHS above; what do you get?