MATH 155: PROBLEM SET 5

DUE FEBRUARY 11

1. Let \( Q(s) = \sum_{n=1}^{\infty} \mu(n)^2/n^s \). In what region does \( Q \) converge absolutely? Does \( Q \) have an Euler product; in what region does that converge absolutely? Can you express \( Q \) in terms of the Riemann zeta-function? Analytically continue \((s-1)Q(s)\) to as large a region as you can.

2. Using Euler-McLaurin or otherwise prove that for \( \sigma > 1 \) we have
\[
\zeta(s) = \frac{s}{s-1} - \frac{1}{2} + \frac{s}{12} + s(s+1) \int_{1-}^{\infty} \frac{(\{y\}^2/2 - \{y\}/2 + 1/12)}{y^{s+2}} \, dy.
\]
Use the RHS to obtain a definition for \( \zeta(s) \) in \( \sigma > -1 \) (and \( s \neq 1 \)). What is \( \zeta(0) \)? Carry out the Euler-McLaurin process one more time to obtain a definition of \( \zeta(s) \) for \( \sigma > -2 \) (and \( s \neq 1 \)). Determine \( \zeta(-1) \).

3. In this problem put \( \sigma = 1 + 1/\log x \). Show that
\[
\sum_{p\leq x} \log \left(1 - \frac{1}{p^\sigma}\right)^{-1} = \sum_{p\leq x} \frac{1}{p^\sigma} + O\left(\frac{1}{x}\right) = \int_{1-}^{\infty} e^{-t} \, dt + O\left(\frac{1}{\log x}\right).
\]
Hint: use that \( \sum_{p\leq z} (\log p)/p = \log z + O(1) \).

4. Again put \( \sigma = 1 + 1/\log x \). Show that
\[
\sum_{p\leq x} \left(\log \left(1 - \frac{1}{p^\sigma}\right)^{-1} - \log \left(1 - \frac{1}{p}\right)^{-1}\right) = -\int_{0}^{1} \frac{1-e^{-t}}{t} \, dt + O\left(\frac{1}{\log x}\right).
\]
This may be a little challenging; one hint is to use the Taylor series for \( \log(1-t)^{-1} \).

5. Using problems 4 and 5 and information about \( \log \zeta(\sigma) \) from class, establish that
\[
\sum_{p\leq x} \left(1 - \frac{1}{p}\right)^{-1} = \log \log x + \int_{0}^{1} \frac{1-e^{-t}}{t} \, dt - \int_{1}^{\infty} \frac{e^{-t}}{t} \, dt + O\left(\frac{1}{\log x}\right).
\]
Use a computer to evaluate the constant in the RHS above; what do you get?