MATH 155: PROBLEM SET 3

Due January 28

1. Prove the identity from class

$$\mu(n)\log n = -\sum_{ab=n}\mu(a)\Lambda(b).$$

2. Let f and g be two multiplicative functions. Show that $f*g$ is also multiplicative. A function is called completely multiplicative if it satisfies $f(mn) = f(m)f(n)$ for all m and n. Is the convolution of two completely multiplicative functions necessarily completely multiplicative?

3. Prove that

$$2^{\omega(n)} = \sum_{d^2m=n}\mu(d)d(m).$$

(Hint: One way to check such identities is to prove them for prime powers, and then use multiplicativity.) Prove that

$$\sum_{n\le x} 2^{\omega(n)} = \frac{6}{\pi^2}x\log x + cx + O(\sqrt{x}\log x),$$

for some constant c. You may use without proof that $\zeta(2) = \pi^2/6$.

4. (a) Prove that

$$\sum_{\substack{m\le x\\ n\le x\\ (m,n)=1}} 1 = -1 + 2\sum_{n\le x}\phi(n).$$

(Hint: either m is bigger than n or n is bigger than m.)

(b) Prove that

$$\sum_{n\le x}\phi(n) = \frac{3}{\pi^2}x^2 + O(x\log x).$$

(c) Conclude that the probability that two random numbers are coprime is $6/\pi^2$.

5. Show that

$$\frac{n}{\phi(n)} = \sum_{d|n}\frac{\mu(d)^2}{\phi(d)}.$$

Show that

$$\sum_{n\le x}\frac{n}{\phi(n)} = cx + O(\log x),$$

for a constant c. Show that $c = \zeta(2)\zeta(3)/\zeta(6)$.