MATH 155: PROBLEM SET 2

Due January 21

1. Let $d_x$ denote the least common multiple of the integers not exceeding $x$. Show that

$$\binom{2n}{n} = \prod_{k=1}^{\infty} d_{2n/k}^{-1}.$$ 

What can you say about the size of $d_x$?

2. From the knowledge that

$$\sum_{n \leq x} \Lambda(n) = \log x + O(1),$$

explain why $\lim_{x \to \infty} \psi(x)/x$, if it exists, must equal 1.

3. Let $d_k(n)$ denote the number of ordered $k$-tuples of positive integers $(d_1, \ldots, d_k)$ with $d_1 \cdots d_k = n$. (Thus $d_2(n) = d(n)$ is the number of divisors of $n$).

(i) Show that

$$d_k(n) = \sum_{ab=n} d_k(b).$$

(ii) Using the hyperbola method, and induction, show that there is a polynomial $P(z)$ of degree $k - 1$ and leading coefficient $1/(k - 1)!$ such that

$$\sum_{n \leq x} d_k(n) = xP_k(\log x) + O(x^{1-1/k}(\log x)^{k-2}).$$

Hint: You may want to choose your parameters $A$ and $B$ carefully.

4. (a). Show that

$$\sum_{pq \leq x} \frac{1}{pq} = \left( \sum_{p \leq x} \frac{1}{p} \right)^2 - 2 \sum_{p \leq \sqrt{x}} \frac{1}{p} \sum_{x/p < q \leq x} \frac{1}{q} - \left( \sum_{\sqrt{x} < p \leq x} \frac{1}{p} \right)^2.$$

(b). For $p \leq \sqrt{x}$ show that

$$\sum_{x/p < q \leq x} \frac{1}{q} = \log \log x - \log \log (x/p) + O\left( \frac{1}{\log x} \right) = O\left( \frac{\log p}{\log x} \right).$$

(c). Conclude that

$$\sum_{pq \leq x} \frac{1}{pq} = \left( \sum_{p \leq x} \frac{1}{p} \right)^2 + O(1).$$
(d). Show that the variance discussed in class

$$\sum_{n \leq x} (\omega(n) - (\log \log x + B))^2$$

is asymptotic to $x \log \log x$. Recall here that $B$ is the constant in the asymptotic

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + B + O\left(\frac{1}{\log x}\right).$$