MATH 155: PROBLEM SET 2

Due January 21

1. Let d_x denote the least common multiple of the integers not exceeding x. Show that
$$\binom{2n}{n} = \prod_{k=1}^{\infty} d_{2n/k}^{(-1)^{k-1}}.$$
What can you say about the size of d_x?

2. From the knowledge that
$$\sum_{n\le x} \frac{\Lambda(n)}{n} = \log x + O(1),$$
explain why $\lim_{x\to\infty} \psi(x)/x$, if it exists, must equal 1.

3. Let $d_k(n)$ denote the number of ordered k-tuples of positive integers $(d_1, \dots, d_k)$ with $d_1 \cdots d_k = n$. (Thus $d_2(n) = d(n)$ is the number of divisors of n).

(i) Show that $d_k(n) = \sum_{ab=n} d_{k-1}(b)$.

(ii) Using the hyperbola method, and induction, show that there is a polynomial $P(z)$ of degree $k-1$ and leading coefficient $1/(k-1)!$ such that
$$\sum_{n\le x} d_k(n) = xP_k(\log x) + O(x^{1-1/k}(\log x)^{k-2}).$$
Hint: You may want to choose your parameters A and B carefully.

4. (a). Show that
$$\sum_{pq\le x} \frac{1}{pq} = \Big(\sum_{p\le x} \frac{1}{p}\Big)^2 - 2\sum_{p\le\sqrt{x}} \frac{1}{p} \sum_{x/p<q\le x} \frac{1}{q} - \Big(\sum_{\sqrt{x}<p\le x} \frac{1}{p}\Big)^2.$$

(b). For $p \le \sqrt{x}$ show that
$$\sum_{x/p<q\le x} \frac{1}{q} = \log\log x - \log\log(x/p) + O\Big(\frac{1}{\log x}\Big) = O\Big(\frac{\log p}{\log x}\Big).$$

(c). Conclude that
$$\sum_{pq\le x} \frac{1}{pq} = \Big(\sum_{p\le x} \frac{1}{p}\Big)^2 + O(1).$$

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(d). Show that the variance discussed in class

$$\sum_{n\leq x}(\omega(n)-(\log\log x+B))^2$$

is asymptotic to $x\log\log x$. Recall here that B is the constant in the asymptotic

$$\sum_{p\leq x}\frac{1}{p}=\log\log x+B+O\Big(\frac{1}{\log x}\Big).$$