1. Find asymptotic formulae for $\sum_{n \leq x} \log n$ and $\sum_{n \leq x} n^{-1/\pi}$. Your error terms should be of size $O((\log x)/x)$ and $O(x^{-1/\pi})$ respectively.

2. Give an example of a non-negative function $f$ such that $f(x) = o(x^\epsilon)$ for any fixed $\epsilon > 0$, but such that $(\log x)^A = o(f(x))$ for any fixed $A > 0$. We would say colloquially that $f$ grows faster than any power of $\log x$ but slower than any power of $x$.

3. For all large $n$ prove that $\omega(n) \ll \log n/\log \log n$ where $\omega(n)$ counts the number of distinct prime factors of $n$.

4. In class we showed that $N! = C\sqrt{N} \left( \frac{N}{e} \right)^N \left( 1 + O\left( \frac{1}{N} \right) \right)$ for some positive constant $C$. In this (challenging) exercise we shall determine that $C = \sqrt{2\pi}$.

   (a). In the formula $2^{2N} = \sum_{k=0}^{2N} \binom{2N}{k}$ show that the terms are increasing in the range $0 \leq k \leq N$ and decreasing thereafter.

   (b). If $k = N + \ell$ with $|\ell| \leq N^{2/3}$ show that

   $$\binom{2N}{k} = \frac{\sqrt{2}}{C\sqrt{N}} 2^{2N} e^{-\ell^2/N + O(\ell^3/N^2)}.$$

   You may find useful here to recall the Taylor approximation for $\log(1 + x)$ for small values of $x$.

   (c). Using (a) and (b) explain why for large $N$ we must have

   $$2^{2N} \sim \frac{\sqrt{2}}{C\sqrt{N}} 2^{2N} \int_{-\infty}^{\infty} e^{-t^2/N} dt,$$

   and conclude that $C = \sqrt{2\pi}$.

5. This exercise recaps and makes precise the Euler-Maclaurin summation formula that we discussed in class. Be warned that the notation is slightly different from what I used in class, and the one below is more standard. The Bernoulli polynomials $B_k(x)$ for $k \geq 0$ are defined by setting $B_0(x) = 1$ for all $x$, and for $k \geq 1$ we have

   $$\frac{d}{dx} B_k(x) = kB_{k-1}(x)$$
and

\[
\int_0^1 B_k(x) \, dx = 0.
\]

You may check that \( B_1(x) = x - 1/2 \), \( B_2(x) = x^2 - x + 1/6 \) etc; the definition in class differs essentially by a factor of \( k! \).

Using induction on \( K \) establish the Euler-Maclaurin formula

\[
\sum_{a < n \leq b} f(n) = \int_a^b f(x) \, dx + \sum_{k=1}^{K} \left( -1 \right)^k \frac{k!}{k!} \left( B_k(\{b\}) f^{(k-1)}(b) - B_k(\{a\}) f^{(k-1)}(a) \right) - \frac{(-1)^K}{K!} \int_a^b B_K(\{x\}) f^{(K)}(x) \, dx.
\]