

NOTES FOR MATH 155: WINTER 2010

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1. EQUIVALENCE OF PNT WITH SUMS OF THE MÖBIUS FUNCTION

Recall that $\mu(n) = 0$ unless n is square-free, and then $\mu(n) = (-1)^{\omega(n)}$ where $\omega(n)$ is the number of prime factors of n . Let

$$M(x) = \sum_{n \leq x} \mu(n).$$

We shall now show that the prime number theorem is equivalent to the estimate $M(x) = o(x)$.

1.1. $M(x) = o(x) \implies$ **PNT**. Suppose first that $M(x) = o(x)$ and we now want to deduce that $\psi(x) \sim x$. Recall that the hypothesis means that for any $\epsilon > 0$, if x is sufficiently large then $|M(x)| \leq \epsilon x$.

We shall make use of the following asymptotics from class:

$$\sum_{n \leq z} \log n = z \log z - z + O(\log z),$$

$$\sum_{n \leq z} d(n) = z \log z + (2\gamma - 1)z + O(\sqrt{z}).$$

Set $b(n) = \log n - d(n) + 2\gamma$. From the above estimates we have that

$$(1) \quad \sum_{n \leq z} b(n) = O(\sqrt{z}).$$

Also by the triangle inequality we see easily that

$$(2) \quad \sum_{n \leq z} |b(n)| \ll z \log z.$$

Next note that $(\mu * b)(n) = (\mu * \log)(n) - (\mu * d)(n) + 2\gamma(\mu * 1)(n) = \Lambda(n) - 1 + 2\gamma\delta(n)$ where $\delta(n) = 1$ if $n = 1$ and 0 otherwise. Thus we

find that

$$\begin{aligned}\psi(x) - [x] + 2\gamma &= \sum_{n \leq x} (\Lambda(n) - 1 + 2\gamma\delta(n)) \\ &= \sum_{n \leq x} (\mu * b)(n) \\ &= \sum_{rs \leq x} \mu(r)b(s).\end{aligned}$$

We now estimate the RHS of the sum above by using the hyperbola method; we would like to show that this is $o(x)$ which would establish PNT. Suppose $RS = x$ and then there are two cases $s \leq S$, and $r \leq x/s$; and $r \leq R$ and $S < s \leq x/r$. We shall take $S = 1/\epsilon$ and we suppose that x is so large that x/S is also large, and for all $z > x/S$ we have $|M(z)| \leq \epsilon z$. Now let us examine the first case: we have

$$\left| \sum_{s \leq S} b(s) \sum_{r \leq x/s} \mu(r) \right| \leq \epsilon x \sum_{s \leq S} |b(s)|/s.$$

Now by (2) and partial summation we see that $\sum_{n \leq z} |b(n)|/n \ll (\log z)^2$ (check!), and therefore the above is

$$\leq C\epsilon(\log S)^2 x \leq \sqrt{\epsilon}x$$

where C is some constant, and the last inequality follows if ϵ is small enough.

Now consider the second case in the hyperbola method. These terms give, by using (1)

$$\left| \sum_{r \leq R} \mu(r) \sum_{S < s \leq x/r} b(s) \right| = O\left(\sum_{r \leq R} |\mu(r)|\sqrt{xr} \right) = O(\sqrt{R}\sqrt{x}) = O(\sqrt{\epsilon}x),$$

since $R = x/S = \epsilon x$.

Thus we have shown that

$$\psi(x) - x + O(1) \ll \sqrt{\epsilon}x.$$

Since ϵ was an arbitrary positive number, it follows that $\psi(x) \sim x$.

1.2. PNT $\implies M(x) = o(x)$. Now let us consider the reverse implication, assuming that $\psi(x) = x + o(x)$. We look at

$$M(x) \log x = \sum_{n \leq x} \mu(n) \log n + \sum_{n \leq x} \mu(n) \log x/n.$$

Note that (prove!)

$$\sum_{n \leq x} \log x/n = O(x),$$

and so

$$M(x) \log x = \sum_{n \leq x} \mu(n) \log n + O(x).$$

Next observe that (again check this!)

$$\mu(n) \log n = - \sum_{ab=n} \mu(a) \Lambda(b).$$

Therefore

$$\begin{aligned} M(x) \log x &= - \sum_{a \leq x} \mu(a) \sum_{b \leq x/a} \Lambda(b) + O(x) \\ &= - \sum_{a \leq x} \mu(a) \left(\psi(x/a) - x/a \right) - \sum_{a \leq x} \mu(a) (x/a) + O(x). \end{aligned}$$

Consider the second term in the RHS above. Since $(\mu * 1)(n) = \delta(n)$ we have

$$\begin{aligned} 1 &= \sum_{n \leq x} \delta(n) = \sum_{n \leq x} (\mu * 1)(n) = \sum_{a \leq x} \mu(a) [x/a] \\ &= \sum_{a \leq x} \mu(a) \left(\frac{x}{a} + O(1) \right) = x \sum_{a \leq x} \mu(a)/a + O(x). \end{aligned}$$

Thus the second term is $O(x)$, and in passing we have also established that

$$\left| \sum_{a \leq x} \frac{\mu(a)}{a} \right| = O(1).$$

Summarizing our work so far, we have

$$|M(x)| \log x \leq \sum_{a \leq x} \left| \psi(x/a) - x/a \right| + O(x).$$

Now we use the PNT in the form $\psi(x) \sim x$. This means that for any $\epsilon > 0$, if z is sufficiently large (say $z > Z$) then $|\psi(z) - z| \leq \epsilon z$. We use this in the sum above when $a \leq x/Z$ (so that $x/a > Z$), and use the Chebyhev bound $\psi(z) = O(z)$ for larger values of a . In this manner we see that

$$\begin{aligned} \sum_{a \leq x} \left| \psi(x/a) - x/a \right| &\leq \sum_{a \leq x/Z} \epsilon x/a + O\left(\sum_{x/Z \leq a \leq x} x/a \right) \\ &\leq \epsilon x \log x + O(x \log Z) \leq 2\epsilon x \log x, \end{aligned}$$

provided x is sufficiently large (so that $x \log Z$ is small compared to $x \log x$). Thus we conclude that $|M(x)| \log x \leq 2\epsilon x \log x + O(x)$, and

since ϵ is an arbitrary positive number we conclude that $M(x) = o(x)$ as needed.