SOLUTIONS FOR THE MIDTERM

KANNAN SOUNDARARAJAN

1. Suppose a number $n \le x$ has a prime factor bigger than $\sqrt{x}$. This prime factor must be unique, and we can then write n uniquely as pm where $p > \sqrt{x}$ is prime, and $m = n/p \le x/p$. Thus the number we want is

$$\sum_{\sqrt{x}<p\le x} \sum_{m\le x/p} 1 = \sum_{\sqrt{x}<p\le x} \Big(\frac{x}{p} + O(1)\Big).$$

The error term above is at most $O(\pi(x)) = O(x/\log x)$ by using a Chebyshev bound for $\pi(x)$. To evaluate the main term, recall from class that

$$\sum_{p\le z} \frac{1}{p} = \log\log z + B + O\Big(\frac{1}{\log z}\Big)$$

for a constant B. Using this with $z = x$ and $z = \sqrt{x}$ and subtracting we find that

$$\sum_{\sqrt{x}\le p\le x} \frac{1}{p} = \log\log x - \log\log\sqrt{x} + O\Big(\frac{1}{\log x}\Big) = \log 2 + O\Big(\frac{1}{\log x}\Big).$$

So the number of such integers is $x\log 2 + O(x/\log x)$.

2. Note that $\sigma(n) = \sum_{d|n} d = \sum_{ab=n} b$. Thus

$$\sum_{n\le x} \sigma(n) = \sum_{a\le x}\sum_{b\le x/a} b = \sum_{a\le x}\Big(\frac{1}{2}\frac{x^2}{a^2} + O\Big(\frac{x}{a}\Big)\Big) = \frac{x^2}{2}\sum_{a\le x}\frac{1}{a^2} + O(x\log x).$$

Now

$$\sum_{a\le x}\frac{1}{a^2} = \zeta(2) + O\Big(\frac{1}{x}\Big) = \frac{\pi^2}{6} + O\Big(\frac{1}{x}\Big).$$

Thus

$$\sum_{n\le x}\sigma(n) = \frac{\pi^2}{12}x^2 + O(x\log x).$$

This solution looks very easy, but the problem has a pitfall: if you had summed the other way

$$\sum_{b\le x} b \sum_{a\le x/b} 1 = \sum_{b\le x} b\Big(\frac{x}{b} + O(1)\Big),$$

then you only get that the sum is $O(x^2)$. But if you fell into this trap, you could use the hyperbola method to get out, which when you optimize would lead to the solution above.

3. Exchanging the sums over p and n we find that

$$\sum_{n\le x} f(n) = \sum_{p\le x} (\log p)^2 \Big(\frac{x}{p} + O(1)\Big).$$

The error term above is

$$O\Big(\sum_{p\le x} (\log p)^2\Big) = O\Big(\log x\vartheta(x)\Big) = O(x\log x)$$

by the Chebyshev bounds. The main term is, by partial summation,

$$x\int_1^x (\log t) d\Big(\sum_{p\le t}\frac{\log p}{p}\Big) = x\Big[(\log t)\sum_{p\le t}\frac{\log p}{p}\Big]_1^x - x\int_1^x \frac{dt}{t}\sum_{p\le t}\frac{\log p}{p}.$$

Now recall from class that

$$\sum_{p\le z}\frac{\log p}{p} = \log z + O(1).$$

Using this above our main term becomes

$$x(\log x)^2 + O(x\log x) - x\int_1^x \frac{(\log t + O(1)}{t}dt = \frac{1}{2}x(\log x)^2 + O(x\log x).$$

We conclude that

$$\sum_{n\le x} f(n) = \frac{1}{2}x(\log x)^2 + O(x\log x).$$

4. Consider the function $g(d) = 0$ unless $d = k^3$ is a perfect cube, in which case set $g(d) = \mu(k)$. Note that g is a multiplicative function, and we wish to show that $(1 * g)(n)$ equals 1 if n is cube-free and zero otherwise. By multiplicativity it suffices to check this in the case of prime powers p^ℓ. Note that $g(1) = 1$, $g(p) = g(p^2) = 0$, $g(p^3) = -1$, $g(p^k) = 0$ for $k \ge 4$. Thus $(1 * g)(p^\ell) = 1$ if $\ell = 0$, 1 or 2, and equals 0 if $\ell \ge 3$; in other words it equals 1 if p^ℓ is cube-free and zero otherwise.

Or instead of doing the above, recall the treatment in class for square-free numbers. Any number n may be written uniquely as $n = a^3 b$ where b is cube-free. Then the condition that n is cube-free amounts to a being 1, and this we can identify using $\sum_{d|a}\mu(d)$.

From the above, the number of cube-free integers up to x equals

$$\sum_{n\le x}\sum_{d^3|n}\mu(d) = \sum_{d\le x^{\frac{1}{3}}}\mu(d)\Big(\frac{x}{d^3} + O(1)\Big).$$

The error term above is $O(x^{1/3})$ while the main term equals

$$x\Big(\sum_{d=1}^{\infty}\frac{\mu(d)}{d^3}+O\Big(\frac{1}{x^{2/3}}\Big)=\frac{x}{\zeta(3)}+O(x^{\frac{1}{3}}).$$

Thus the number of cube-free integers up to x equals $x/\zeta(3)+O(x^{\frac{1}{3}})$.