Math 155: Homework 2

January 25, 2010

1. We begin with a

Lemma 0.1. *Suppose k is an integer. Then*

$$d_k = \begin{cases} pd_{k-1} & \text{if } k = p^m, \\ d_{k-1} & \text{else.} \end{cases}$$

That is to say,

$$\log(d_x) = \sum_{n \le x} \Lambda(n) = \psi(x).$$

Proof (Sketch): If k is composite, then suppose $k = p_1^{e_1} \cdots p_m^{e_m}$. Then each of $p_i^{e_i} < k-1$, and hence $p_i^{e_i} | d_{k-1}$. Then we must have $k | d_{k-1}$, hence $d_k = d_{k-1}$. On the other hand, suppose $k = p^m$. Then we have that $p^{m-1} | d_{k-1}$, but p^m does not. Hence, $d_k = pd_{k-1}$.

Taking the log of the first part of the lemma, we find that $\log(d_k) = \log(d_{k-1}) + \Lambda(k)$. Thus we must of course have $\log(d_x) = \psi(x)$. QED.

We start by writing $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$, and then by writing out a few special cases, one observes that

$$\frac{1}{n!} = \prod_{k \text{ even}} d_{2n/k}^{-1}, \qquad \text{and} \qquad \frac{(2n)!}{n!} = \prod_{k \text{ odd}} d_{2n/k}.$$

The second of these actually follows from the first: let $j = 2k$, then $n! = \prod_{j=1}^{\infty} d_{n/j}$, so that

$$\frac{(2n)!}{n!} = \frac{\prod_{k=1}^{\infty} d_{2n/k}}{\prod_{k \text{ even}} d_{2n/k}} = \prod_{k \text{ odd}} d_{2n/k}.$$

By multiplying these two back together, we get the original statement of the exercise. So it suffices to prove that $n! = \prod_{k=1}^{\infty} d_{n/k}$. Indeed, first observe that $d_{n/k} = 1$ for $k > n$, so we have

$$\log(\prod_{k=1}^{\infty} d_{n/k}) = \sum_{k=1}^{n} \log d_{n/k} = \sum_{k=1}^{n} \sum_{l \leq n/k} \Lambda(l)$$

by the lemma. We can re-write the sum with $lk = m$ to give

$$\log(\prod_{k=1}^{\infty} d_{n/k}) = \sum_{m=1}^{n} \sum_{l|m} \Lambda(l) = \sum_{m=1}^{n} \log m = \log n!,$$

hence collecting our results, we get

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \prod_{k=1}^{\infty} d_{2n/k}^{(-1)^{k-1}}.$$

Lastly, we have $\psi(x) = x + O(xe^{-c\sqrt{\log(x)}})$ from class, so using the lemma again, we in fact have that $d_x = e^x + O(\exp(xe^{-c\sqrt{\log(x)}}))$.

2. We apply the partial summation formula

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - A_{M-1} b_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n$$

with $a_n = 1/n$, and $b_n = \Lambda(n)$. We should have

$$1 = \lim_{x \to \infty} \frac{1}{\log(x)} \sum_{n \leq x} \frac{\Lambda(n)}{n},$$

so we apply partial summation to get

$$1 = \lim_{x \to \infty} \frac{1}{\log x} \left(\frac{\psi(x)}{x} + \sum_{n \leq x-1} \frac{\psi(n)}{n(n+1)} \right).$$

We assume that $\lim_{x \to \infty} \psi(x)/x = C$. Let M be sufficiently large so that $|C - \psi(x)/x| \leq \varepsilon$ for all $x \geq M$. Then we have

$$1 = \lim_{x \to \infty} \frac{C}{\log(x)} + \lim_{x \to \infty} \frac{1}{\log(x)} \left(\sum_{n=1}^{M} \frac{\psi(n)}{n(n+1)} + \sum_{n=M+1}^{x-1} \frac{\psi(n)}{n(n+1)} \right).$$

M is fixed, so $\sum_{n=1}^{M} \frac{\psi(n)}{n(n+1)}$ is a constant, say $K(M)$. Then,

$$1 = \lim_{x \to \infty} \frac{K(M)}{\log(x)} + \lim_{x \to \infty} \frac{1}{\log(x)} \sum_{n=M+1}^{x-1} \frac{\psi(n)}{n(n+1)}$$

$$\leq \lim_{x\to\infty} \frac{1}{\log(x)}(C+\varepsilon)\sum_{n=M+1}^{x-1}\frac{1}{n+1} = \lim_{x\to\infty}\frac{C+\varepsilon}{\log(x)}(\log(x)-\log(M)+O(1)) = C+\varepsilon.$$

As ε was arbitrary, this yields a contradiction if $C < 1$. The same computation with $C-\varepsilon$ will yield a contradiction if $C > 1$. Thus, the only possibility is that $C = 1$, as was to be shown.

3. (i) Let $S_k(n)$ be the set of ordered tuples $\{(d_1,\ldots,d_k)|d_1\cdots d_k = n\}$. Then

$$S_n = \coprod_{m|n}\{(m,e_1,\ldots,e_{k-1})|me_1\cdots e_{k-1}=n\},$$

the disjoint union. This gives

$$d_k(n) = |S_k(n)| = \sum_{m|n} d_{k-1}(n/m) = \sum_{ab=n} d_{k-1}(b).$$

(ii) This part is an extended computation with induction, hyperbola method, and summation by parts. Assume the exercise is true for $k-1$. Then

$$\begin{aligned}\sum_{n\leq x} d_k(n) &= \sum_{n\leq x}\sum_{ab=n} d_{k-1}(b) = \sum_{ab\leq x} d_{k-1}(b)\\ &= \sum_{\substack{a\leq A\\ b\leq x/a}} d_{k-1}(b) + \sum_{\substack{b\leq B\\ a\leq x/b}} d_{k-1}(b) - \sum_{\substack{a\leq A\\ b\leq B}} d_{k-1}(b)\\ &= \sum_{a\leq A}\sum_{b\leq x/a} d_{k-1}(b) + \sum_{b\leq B}\frac{x}{b}d_{k-1}(b) - A\sum_{b\leq B} d_{k-1}(b).\end{aligned}$$

We use the induction hypothesis to treat the first sum.

$$\sum_{a\leq A}\sum_{b\leq x/a} d_{k-1}(b) = \sum_{a\leq A}\frac{x}{a}P_{k-1}(\log(x/a)) + O((x/a)^{1-1/(k-1)}(\log(x/a))^{k-3}).$$

We deal with only the leading order term of this sum, since we may define the lower-order coefficients of $P_k(z)$ however we like. We evaluate the sum

$$\frac{x}{(k-2)!}\sum_{a\leq A}\frac{\log^{k-2}(x/a)}{a}$$

via Euler-Maclaurin. The formula for $K = 1$ is

$$\sum_{a\leq n\leq b} f(n) = \int_a^b f(x)\,dx - (B_1(\{b\})f(b) - B_1(\{a\})f(a)) + \int_a^b B_1(\{x\})f'(x)\,dx.$$

So we have that

$$\sum_{a\leq A}\frac{\log^{k-2}(x/a)}{a} = \int_{1/2}^{A}\frac{\log^{k-2}(x/u)}{u}\,du - \frac{\log^{k-2}(x/A)}{A}(\{A\}-\frac{1}{2}) - \int_{1/2}^{A}(\{u\}-\frac{1}{2})\frac{d}{du}\left(\frac{\log^{k-2}(x/u)}{u}\right)du$$

$$= -\frac{\log^{k-1}(x/A)}{k-1} + \frac{\log^{k-1}(2x)}{k-1} + O\left(\frac{\log^{k-2}(x/A)}{A}\right). \tag{1}$$

Now we move on to the second sum. We apply summation by parts to get

$$\sum_{b \le B} \frac{1}{b} d_{k-1}(b) = \frac{1}{B} \sum_{b \le B} d_{k-1}(b) + \sum_{b \le B-1} \frac{1}{b(b+1)} \sum_{i \le b} d_{k-1}(i)$$

$$= P_{k-1}(\log(B)) + \sum_{b \le B-1} \frac{P_{k-1}(\log(b))}{b+1} + O(B^{-1/(k-1)} \log^{k-2}(B)).$$

Again, the lower order terms in $P_{k-1}(\log(b))$ in the above sum will only contribute to the lower order terms in $P_k(z)$, which we may define however we like, so we only treat the top order term. We want to evaluate $\sum_{b \le B} \frac{\log^{k-2}(\log(b))}{b+1}$ by Euler-Maclaurin, however, this summand is not one we know how to compute the integral of easily. We can make a quick fix however:

$$\sum_{b \le B} \frac{\log^{k-2}(b)}{b} - \sum_{b \le B} \frac{\log^{k-2}(b)}{b+1} = \sum_{b \le B} \frac{\log^{k-2}(b)}{b(b+1)} = C + O\left(\frac{1}{B^{1-\varepsilon}}\right)$$

for some explicit constant C, and some small ε. Specifically, we need $1 - \varepsilon > \frac{1}{k-1}$ (the error term here is very crude, but sufficient). We thus have by Euler-Maclaurin

$$\sum_{b \le B} \frac{\log^{k-2}(b)}{b+1} = \frac{1}{k-1} \log^{k-1}(B) + C + O\left(\frac{1}{B^{1-\varepsilon}}\right) \tag{2}$$

.

The last sum is small. It does not contribute to the highest order term:

$$\sum_{b \le B} d_{k-1}(b) = BP_{k-1}(\log(B)) + O(B^{1-1/(k-1)} \log^{k-3}(B)) \tag{3}$$

Pulling results (1),(2) and (3) together, we have

$$\sum_{n \le x} d_k(n) = \sum_{a \le A} \sum_{b \le x/a} d_{k-1}(b) + \sum_{b \le B} \frac{x}{b} d_{k-1}(b) - A \sum_{b \le B} d_{k-1}(b)$$

$$= \frac{x}{(k-2)!} \left(-\frac{\log^{k-1}(x/A)}{k-1} + \frac{\log^{k-1}(2x)}{k-1} + (\text{lower powers of} \log(x/A)) + O\left(\frac{\log^{k-2}(x/A)}{A}\right)\right)$$

$$+ \sum_{a \le A} O((x/a)^{1-1/(k-1)} (\log(x/a))^{k-3})$$

$$+ x \left(\frac{1}{(k-1)!} \log^{k-1}(B) + (\text{lower powers of} \log(B)) + O(B^{-1/(k-1)} \log^{k-2}(B)) \right)$$

$$- ABP_{k-1}(\log(B)) + O(AB^{1-1/(k-1)} \log^{k-3}(B)).$$

We will pick A and B in a moment, but first use $AB = x$ to clean this up a bit:

$$\sum_{n\le x} d_k(n) = \frac{x}{(k-1)!}(\log^{k-1}(x) - \log^{k-1}(x/A) + \log^{k-1}(B)) + (\text{lower powers of} \log(x/A)) + (\text{lower powers of} \log(B))$$

$$+O\left(\frac{x\log^{k-2}(x/A)}{A}\right) + \sum_{a\le A} O((x/a)^{1-1/(k-1)}(\log(x/a))^{k-3}) + O(xB^{-1/(k-1)}\log^{k-2}(B))$$

$$-xP_{k-1}(\log(B)) + O(xB^{-1/(k-1)}\log^{k-3}(B)).$$

Now, we first of all need to have $\log^{k-1}(x/A) = \log^{k-1}(B)$. We guess that $A = x^\alpha$, and $B = x^{1-\alpha}$ for some $0 \le \alpha \le 1$. But then we have $\log^{k-1}(x/A) = \log^{k-1}(B)$ for any such choice of α. So this choice only affects the error terms. We treat them individually. To get the error we want from $O\left(\frac{x\log^{k-2}(x/A)}{A}\right)$, we must take $\alpha = 1/k$. This gives

$$O\left(\frac{x\log^{k-2}(x/A)}{A}\right) = O(x^{1-1/k}\log^{k-2}(x)),$$

as desired. Next, we have by Euler-Maclaurin again

$$\sum_{a\le A} O((x/a)^{1-1/(k-1)}(\log(x/a))^{k-3}) = O(x^{1-1/(k-1)}A^{1/(k-1)}\log^{k-2}(x/A)),$$

which for our choice of α reduces to $O(x^{1-1/k}\log^{k-2}(x))$. Finally,

$$O(xB^{-1/(k-1)}\log^{k-2}(B)) = O(x(x^{1-1/k})^{-1/(k-1)}\log^{k-2}(B)) = O(x^{1-1/k}\log^{k-2}(x)),$$

as desired. Thus we have

$$\sum_{n\le x} d_k(n) = \frac{x}{(k-1)!}\log^{k-1}(x) + (\text{lower powers of} \log(x/A)) + (\text{lower powers of} \log(B)) + O(x^{1-1/k}\log^{k-2}(x)).$$

Finally, defining the lower order coefficients of $P_k(z)$ in whichever way is specified by the above, we arrive at our desired result.

4. (a) This part amounts to applying the hyperbola method and then doing some acrobatics with the summations. Let $A = B = \sqrt{x}$ and apply the hyperbola method:

$$
\begin{aligned}
\sum_{pq\le x}\frac{1}{pq} &= \sum_{p\le\sqrt{x}}\sum_{q\le x/p}\frac{1}{pq}+\sum_{q\le\sqrt{x}}\sum_{p\le x/q}\frac{1}{pq}-\sum_{p\le\sqrt{x}}\sum_{q\le\sqrt{x}}\frac{1}{pq}\\
&= \sum_{p\le\sqrt{x}}\frac{1}{p}\sum_{q\le x/p}\frac{1}{q}+\sum_{q\le\sqrt{x}}\frac{1}{q}\sum_{p\le x/q}\frac{1}{p}-\sum_{p\le\sqrt{x}}\frac{1}{p}\sum_{q\le\sqrt{x}}\frac{1}{q}\\
&= 2\sum_{p\le\sqrt{x}}\frac{1}{p}\sum_{q\le x/p}\frac{1}{q}-\left(\sum_{p\le\sqrt{x}}\frac{1}{p}\right)^2\\
&= 2\sum_{p\le\sqrt{x}}\frac{1}{p}\sum_{q\le x/p}\frac{1}{q}-\left(\sum_{p\le x}\frac{1}{p}-\sum_{\sqrt{x}<p\le x}\frac{1}{p}\right)^2\\
&= 2\sum_{p\le\sqrt{x}}\frac{1}{p}\sum_{q\le x/p}\frac{1}{q}-\left(\left(\sum_{p\le x}\frac{1}{p}\right)^2-2\sum_{p\le x}\frac{1}{p}\sum_{\sqrt{x}<p\le x}\frac{1}{p}+\left(\sum_{\sqrt{x}<p\le x}\frac{1}{p}\right)^2\right)\\
&= 2\sum_{p\le\sqrt{x}}\frac{1}{p}\sum_{q\le x/p}\frac{1}{q}+2\sum_{p\le x}\frac{1}{p}\left(\sum_{p\le x}\frac{1}{p}-\sum_{p\le\sqrt{x}}\frac{1}{p}\right)-\left(\sum_{p\le x}\frac{1}{p}\right)^2-\left(\sum_{\sqrt{x}<p\le x}\frac{1}{p}\right)^2\\
&= -2\left(\sum_{p\le\sqrt{x}}\frac{1}{p}\sum_{p\le x}\frac{1}{p}-\sum_{p\le\sqrt{x}}\frac{1}{p}\sum_{q\le x/p}\frac{1}{q}\right)+2\sum_{p\le x}\frac{1}{p}\sum_{p\le x}\frac{1}{p}-\left(\sum_{p\le x}\frac{1}{p}\right)^2-\left(\sum_{\sqrt{x}<p\le x}\frac{1}{p}\right)^2\\
&= -2\left(\sum_{p\le\sqrt{x}}\frac{1}{p}\sum_{x/p<q\le x}\frac{1}{q}\right)+\left(\sum_{p\le x}\frac{1}{p}\right)^2-\left(\sum_{\sqrt{x}<p\le x}\frac{1}{p}\right)^2.
\end{aligned}
$$

(b) The sum is over primes q, so we use the estimates from class:

$$
\begin{aligned}
\sum_{x/p<q\le x}\frac{1}{q} &= \sum_{q\le x}\frac{1}{q}-\sum_{q\le x/p}\frac{1}{q}\\
&= \left(\log\log x+B+O\left(\frac{1}{\log x}\right)\right)-\left(\log\log(x/p)+B+O\left(\frac{1}{\log(x/p)}\right)\right)\\
&= \log\log x-\log\log(x/p)+O\left(\frac{1}{\log x}\right)\\
&= \log\left(\frac{\log x}{\log x-\log p}\right)+O\left(\frac{1}{\log x}\right)\\
&= -\log(1-\frac{\log p}{\log x})+O\left(\frac{1}{\log x}\right)\\
&= O\left(\frac{\log p}{\log x}\right),
\end{aligned}
$$

where in the third line we have used $p \leq \sqrt{x}$, which implies $\frac{1}{\log(x/p)} \leq \frac{1}{\log(x)}$, and in the final line we have used the Taylor expansion for log.

(c) Observe that we can apply (b) with $p = \sqrt{x}$ to obtain

$$\sum_{\sqrt{x}<q\leq x} \frac{1}{q} = O\left(\frac{\log(\sqrt{x})}{\log(x)}\right) = O(1).$$

So we now have combining part (a) and (b)

$$\begin{aligned}
\sum_{pq\leq x} \frac{1}{pq} &= \left(\sum_{p\leq x} \frac{1}{p}\right)^2 - 2\sum_{p\leq\sqrt{x}} \frac{1}{p} \sum_{x/p<q\leq x} \frac{1}{q} - \left(\sum_{\sqrt{x}<p\leq x} \frac{1}{p}\right)^2 \\
&= \left(\sum_{p\leq x} \frac{1}{p}\right)^2 + O\left(\sum_{p\leq\sqrt{x}} \frac{1}{p}\frac{\log p}{\log x}\right) + O(1) \\
&= \left(\sum_{p\leq x} \frac{1}{p}\right)^2 + O\left(\frac{1}{\log x} O(\log\sqrt{x})\right) + O(1) \\
&= \left(\sum_{p\leq x} \frac{1}{p}\right)^2 + O(1),
\end{aligned}$$

as was to be shown. Note that the third line follows from Tchebychev type estimates from class.

(d) We begin with a

Lemma 0.2. *Let f be an arithmetic function. The we have*

$$\sum_{n\leq x}\sum_{d|n} f(d) = x\sum_{d\leq x}\frac{f(d)}{d} + O\left(\sum_{d\leq x}|f(d)|\right).$$

Proof: Swap the order or summation, and apply the estimate $\lfloor y \rfloor = y + O(1)$,

$$\sum_{n\leq x}\sum_{d|n} f(d) = \sum_{d\leq x} f(d) \sum_{\substack{n\leq d \\ d|n}} 1 = \sum_{d\leq x} f(d)\lfloor\frac{x}{d}\rfloor = x\sum_{d\leq x}\frac{f(d)}{d} + O\left(\sum_{d\leq x}|f(d)|\right).$$

QED.

Now, continuing with the proof, we expand the expression:

$$
\begin{aligned}
\sum_{n\le x}(\omega(n)-\log\log x-B)^2 &= \sum_{n\le x}\omega(n)^2+\sum_{n\le x}(\log\log x)^2+\sum_{n\le x}B^2 \\
&\quad -2\sum_{n\le x}\omega(n)\log\log x-2\sum_{n\le x}B\omega(n)+2\sum_{n\le x}B\log\log x. \qquad (4)
\end{aligned}
$$

First, let's do the easy ones:

$$\sum_{n\le x}(\log\log x)^2 = x(\log\log x)^2$$

$$\sum_{n\le x}B^2 = O(x)$$

$$2\sum_{n\le x}B\log\log x = 2Bx\log\log x.$$

Now, we need to evaluate $\sum_{n\le x}\omega(n)$. For this, we define

$$\chi(d)=\begin{cases}1 & \text{if } d \text{ is prime,}\\ 0 & \text{else.}\end{cases}$$

Now, we have by this, and the lemma,

$$
\begin{aligned}
\sum_{n\le x}\omega(n) &= \sum_{n\le x}\sum_{p|n}1=\sum_{n\le x}\sum_{d|n}\chi(d)=x\sum_{d\le x}\frac{\chi(d)}{d}+O(\pi(x)) \\
&= x\sum_{p\le x}\frac{1}{p}+O\left(\frac{x}{\log(x)}\right)=x\log\log x+Bx+O\left(\frac{x}{\log(x)}\right).
\end{aligned}
$$

So that we can evaluate another two terms of (4):

$$-2\sum_{n\le x}\omega(n)\log\log x=-2x(\log\log x)^2-2Bx\log\log x+O(x)$$

$$-2\sum_{n\le x}B\omega(n)=-2Bx\log\log x+O(x).$$

Finally, we come to estimating the main term $\sum_{n\le x}\omega(n)^2$. Note that by part (c), we have

$$\sum_{pq\le x}\frac{1}{pq}=(\log\log x)^2+2B\log\log x+O(1).$$

Proceeding similarly to the above,

$$\sum_{n\le x}\omega(n)^2 = \sum_{n\le x}\left(\sum_{d_1|n}\chi(d_1)\right)\left(\sum_{d_2|n}\chi(d_2)\right)$$

$$= \sum_{\substack{n\le x\\ d_1=d_2}}\left(\sum_{d_1|n}\chi(d_1)\right)\left(\sum_{d_2|n}\chi(d_2)\right) + \sum_{\substack{n\le x\\ d_1\ne d_2}}\left(\sum_{d_1|n}\chi(d_1)\right)\left(\sum_{d_2|n}\chi(d_2)\right)$$

$$= \sum_{n\le x}\omega(n) + \sum_{n\le x}\sum_{d_1 d_2|n}\chi(d_1)\chi(d_2)$$

$$= \sum_{n\le x}\omega(n) + x\sum_{pq\le x}\frac{1}{pq} + O(1) = x(\log\log x)^2 + (2B+1)x\log\log x + O(x)$$

by pulling together our previous results.

Adding together the six summands of (4), we have many cancelations, and arrive at our desired result:

$$\sum_{n\le x}(\omega(n) - \log\log x - B)^2 = x\log\log x + O(x).$$

Hence

$$\sum_{n\le x}(\omega(n) - \log\log x - B)^2 \sim x\log\log x.$$