Math 155: Homework 1

January 21, 2010

1. We solve this problem by applying Euler-Maclaurin as it appears in exercise 5 for $K = 1$:

$$\sum_{a \le n \le b} f(n) = \int_a^b f(x)\,dx - (B_1(\{b\})f(b) - B_1(\{a\})f(a)) + \int_a^b B_1(\{x\})f'(x)\,dx.$$

Letting $f(u) = \log(u)/u$, this becomes

$$\begin{aligned}\sum_{n \le x} \frac{\log(n)}{n} &= \int_1^x \frac{\log(u)}{u}\,du - ((\{x\} - \frac{1}{2})\frac{\log(x)}{x} - (\{1\} - \frac{1}{2})\frac{\log(1)}{1}) + \int_1^x (\{u\} - \frac{1}{2})\frac{d}{du}\left(\frac{\log(u)}{u}\right)du \\ &= \frac{1}{2}\log^2(x) + O\left(\frac{\log(x)}{x}\right) + \int_1^x (\{u\} - \frac{1}{2})\frac{d}{du}\left(\frac{\log(u)}{u}\right)du \\ &= \frac{1}{2}\log^2(x) + O\left(\frac{\log(x)}{x}\right) + O\left(\int_1^x \frac{d}{du}\left(\frac{\log(u)}{u}\right)du\right) \\ &= \frac{1}{2}\log^2(x) + O\left(\frac{\log(x)}{x}\right),\end{aligned}$$

as was to be shown.

Now we do the $n^{-1/\pi}$ case. Apply Euler-Maclaurin again:

$$\begin{aligned}\sum_{n \le x} n^{-1/\pi} &= \int_1^x u^{-1/\pi}\,du - ((\{x\} - \frac{1}{2})x^{-1/\pi} - (\{1\} - \frac{1}{2})1^{-1/\pi}) + \int_1^x (\{u\} - \frac{1}{2})\frac{d}{du}(u^{-1/\pi})\,du \\ &= \frac{1}{1 - \frac{1}{\pi}}\left(x^{1-1/\pi} - 1\right) + O(x^{-1/\pi}) + \frac{1}{2} + O\left(\int_1^x \frac{d}{du}(u^{-1/\pi})\,du\right) \\ &= \frac{1}{1 - \frac{1}{\pi}}x^{1-1/\pi} + \left(\frac{1}{2} - \frac{1}{1 - \frac{1}{\pi}}\right) + O(x^{-1/\pi}),\end{aligned}$$

as was to be shown.

2. The first example which comes to mind for me is

$$f(x) = e^{\sqrt{\log(x)}}.$$

This happens to be the error term you will get when you prove the prime number theorem. Once you have this idea, it's easy to check that it's the right answer:

$$\frac{e^{\sqrt{\log(x)}}}{x^\epsilon} = \frac{e^{\sqrt{\log(x)}}}{e^{\epsilon\log(x)}} = e^{\sqrt{\log(x)}-\epsilon\log(x)},$$

and $\sqrt{\log(x)} - \epsilon\log(x) \to -\infty$ as $x \to \infty$ for any ϵ, hence $e^{\sqrt{\log(x)}-\epsilon\log(x)} \to 0$ as $x \to \infty$ for any ϵ. Hence $f(x) = o(x^\epsilon)$ for any ϵ. On the other hand,

$$\frac{\log^A(x)}{e^{\sqrt{\log(x)}}} = \frac{e^{\log(\log^A(x))}}{e^{\sqrt{\log(x)}}} = e^{A\log(\log(x))-\sqrt{\log(x)}}.$$

And $A\log(\log(x)) - \sqrt{\log(x)} \to -\infty$ as $x \to \infty$ for any A, so $e^{A\log(\log(x))-\sqrt{\log(x)}} \to 0$ as $x \to \infty$ for any A. So we have $\log^A(x) = o(f(x))$, as was to be shown.

3. Think about the "worst case scenario": numbers of the form $n = \prod_{p\le M} p$. In this exercise, apply the Tchebychev bounds which were mentioned in class: $cM \le \sum_{p\le M}\log(p) \le c'M$, for some c, c', and the resulting bounds $b(x/\log(x)) \le \pi(x) \le b'(x/\log(x))$, for some b, b'. In fact, we have that if n is of the above form, then we must have $\log(n) \ge cM$, and $\log(\log(n)) \le \log(M) + \log(c')$. Thus we have

$$\frac{\log(n)}{\log(\log(n))} \ge \frac{cM}{\log(M)+\log(c')} \ge C\frac{M}{\log(M)},$$

for some other constant C. But, we have $\pi(M) = \omega(n)$, from the way in which M was defined. Then

$$\frac{\log(n)}{\log(\log(n))} \ge C\frac{M}{\log(M)} \ge \frac{C}{b'}\pi(M) = C'\omega(n),$$

as was to be shown.

4. (a) The idea of the proof is via induction, interpreting $\binom{2N}{k}$ as the k-th entry of the $2N$-th line of Pascal's triangle. More explicitly, we have the following recurrence relation:

$$\binom{2N}{k} = \binom{2N-1}{k-1} + \binom{2N-1}{k}.$$

Assume that the $2N-1$-th binomial coefficients are increasing for $0 \le k \le \frac{2N-1}{2}$, and decreasing for $\frac{2N-1}{2} \le k \le 2N-1$. Then for any $0 \le k \le N$ and $N \le j \le 2N$, we have

$$\binom{2N-1}{k-2} < \binom{2N-1}{k} \text{ and } \binom{2N-1}{j-1} > \binom{2N-1}{j+1}.$$

Whence

$$\binom{2N}{k-1} = \binom{2N-1}{k-2} + \binom{2N-1}{k-1} < \binom{2N-1}{k} + \binom{2N-1}{k-1} = \binom{2N}{k}$$

and

$$\binom{2N}{j} = \binom{2N-1}{j-1} + \binom{2N-1}{j} > \binom{2N-1}{j+1} + \binom{2N-1}{j} = \binom{2N}{j+1}$$

(b) First of all, the exercise is incorrect as stated. The correct statement should be

$$\binom{2N}{N+l} = \frac{\sqrt{2}}{C\sqrt{N}} 2^{2N} \exp\left(-\frac{l^2}{N} + O\left(\frac{|l|^3}{N^2}\right) + O\left(\frac{1}{N}\right)\right).$$

This calculation can get very hairy if you aren't careful about the error terms. A nice way to do it is first pull out the part involving the l's:

$$\begin{aligned}
\binom{2N}{N+l} &= \frac{(2N)!}{(N+l)!(N-l)!} \\
&= \frac{(2N)!}{(N!)^2} \prod_{i=1}^{l} \frac{N-i}{N+i} \\
&= \frac{(2N)!}{(N!)^2} \prod_{i=1}^{l} \left(1 + \frac{2i}{N-i}\right)^{-1} \\
&= \frac{(2N)!}{(N!)^2} \exp\left(-\sum_{i=1}^{l} \log\left(1 + \frac{2i}{N-i}\right)\right) \\
&= \frac{(2N)!}{(N!)^2} \exp\left(-\sum_{i=1}^{l} \frac{2i}{N-i} + O\left(\frac{i^2}{N^2}\right)\right) \\
&= \frac{(2N)!}{(N!)^2} \exp\left(-\sum_{i=1}^{l} \frac{2i}{N} + O\left(\frac{|l|^3}{N^2}\right)\right) \\
&= \frac{(2N)!}{(N!)^2} \exp\left(-\frac{l^2}{N} + O\left(\frac{|l|^3}{N^2}\right)\right).
\end{aligned}$$

Where we have used the summation formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ in the last line, and in the fifth line we use the $|l| \leq N^{2/3}$ to ensure the Taylor approximation to $\log(1+x)$ converges.

Now we use Stirling's formula to evaluate $\frac{(2N)!}{(N!)^2}$:

$$\begin{aligned}
\frac{(2N)!}{(N!)^2} &= \frac{C\sqrt{2N}\left(\frac{2N}{e}\right)^{2N}\left(1 + O\left(\frac{1}{N}\right)\right)}{\left(C\sqrt{N}\left(\frac{N}{e}\right)^{N}\left(1 + O\left(\frac{1}{N}\right)\right)\right)^2} \\
&= \frac{\sqrt{2}\,2^{2N}\left(1 + O\left(\frac{1}{N}\right)\right)}{C\sqrt{N}\left(1 + O\left(\frac{1}{N}\right)\right)^2}.
\end{aligned}$$

Note we can take a Taylor expansion of $\frac{1}{1+x}$ to find

$$\frac{1}{1 + O\left(\frac{1}{N}\right)} = 1 + O\left(\frac{1}{N}\right).$$

Thus, we get

$$\frac{(2N)!}{(N!)^2} = \frac{\sqrt{2}\,2^{2N}}{C\sqrt{N}}\left(1 + O\left(\frac{1}{N}\right)\right).$$

Thus, in total, we get

$$\binom{2N}{N+l} = \frac{\sqrt{2}}{C\sqrt{N}} 2^{2N} \exp\left(-\frac{l^2}{N} + O\left(\frac{|l|^3}{N^2}\right) + O\left(\frac{1}{N}\right)\right).$$

(c) Observe that

$$2^{2N} = \sum_{l=-N}^{N} \binom{2N}{N+l} = \sum_{N^{2/3}<|l|\leq N} \binom{2N}{N+l} + \frac{\sqrt{2}}{C\sqrt{N}} 2^{2N} e^{O(1/N)} \sum_{|l|\leq N^{2/3}} \exp\left(-\frac{l^2}{N} + O\left(\frac{|l|^3}{N^2}\right)\right).$$

But in the range $N^{2/3} < |l| \leq N$, we have by part (a) that $\binom{2N}{N+l} \leq \binom{2N}{N^{1/3}} = O(2^{2N^{1/3}})$. So then

$$\sum_{N^{2/3}<|l|\leq N} \binom{2N}{N+l} = O(N^{1/3} 2^{2N^{1/3}}).$$

Furthermore, we also have by similar reasoning to problem 1

$$\sum_{|l|\leq N^{2/3}} e^{-\frac{l^2}{N}} = \int_{-N^{2/3}}^{N^{2/3}} e^{-t^2/N}\, dt + O\left(e^{-N^{1/3}}\right).$$

Thus, we get in total:

$$1 = \frac{1}{2^{2N}} \left(\sum_{N^{2/3}<|l|\leq N} \binom{2N}{N+l} + \frac{\sqrt{2}}{C\sqrt{N}} 2^{2N} e^{O(1/N)} \sum_{|l|\leq N^{2/3}} \exp\left(-\frac{l^2}{N} + O\left(\frac{|l|^3}{N^2}\right)\right)\right)$$

$$= \lim_{N\to\infty} \frac{O(2^{2N^{1/3}})}{2^{2N}} + \lim_{N\to\infty} \frac{1}{2^{2N}} \frac{\sqrt{2}}{C\sqrt{N}} 2^{2N} \int_{-N^{2/3}}^{N^{2/3}} e^{-t^2/N}\, dt$$

$$= \lim_{N\to\infty} \frac{1}{2^{2N}} \frac{\sqrt{2}}{C\sqrt{N}} 2^{2N} \int_{-\infty}^{\infty} e^{-t^2/N}\, dt.$$

Which is exactly

$$2^{2N} \sim \frac{\sqrt{2}}{C\sqrt{N}} 2^{2N} \int_{-\infty}^{\infty} e^{-t^2/N}\, dt.$$

Finally, we evaluate the Gaussian:

$$\int_{-\infty}^{\infty} e^{-t^2/N}\, dt = \sqrt{\pi N},$$

whence

$$2^{2N} \sim \frac{\sqrt{2}\sqrt{\pi N}}{C\sqrt{N}} 2^{2N}$$

Since we must have $2^{2N} \sim 2^{2N}$, we finally get that $C = \sqrt{2\pi}$, as was to be shown.

5. This exercise uses the Riemann–Stieltjes integral, so if you haven't seen that before, or don't remember the details, I suggest you look it up. First, we do the base case. Note that $\lfloor x \rfloor = x - \{x\}$, so:

$$\begin{aligned}
\sum_{a \le n \le b} f(n) &= \int_a^b f(x)\, d\lfloor x \rfloor \\
&= f(b)\lfloor b \rfloor - f(a)\lfloor a \rfloor - \int_a^b \lfloor x \rfloor df(x) \\
&= bf(b) - \{b\}f(b) - af(a) + \{a\}f(a) - \int_a^b xf'(x)\, dx + \int_a^b \{x\}f'(x)\, dx \\
&= -\{b\}f(b) + \{a\}f(a) + \int_a^b f(x)\, dx + \int_a^b \{x\}f'(x)\, dx.
\end{aligned}$$

Then, observe that $B_1(\{x\}) = \{x\} - 1/2$, and $0 = \frac{1}{2}\int_a^b f'(x)\, dx - \frac{1}{2}(f(b) - f(a))$ applied to the above give

$$\sum_{a \le n \le b} f(n) = \int_a^b f(x)\, dx - (B_1(\{b\})f(b) - B_1(\{a\})f(a)) + \int_a^b B_1(\{x\})f'(x)\, dx,$$

which is exactly the $K = 1$ case of Euler-Maclaurin.

For the induction step, we again use the R-S integral and integration by parts,

$$\begin{aligned}
&\frac{(-1)^{K+1}}{(K+1)!}\int_a^b B_{K+1}(\{x\})f^{(K+1)}(x)\, dx \\
= \;&\frac{(-1)^{K+1}}{(K+1)!}\int_a^b B_{K+1}(\{x\})\, df^{(K)}(x) \\
= \;&\frac{(-1)^{K+1}}{(K+1)!}\left(B_{(K+1)}(\{b\})f^{(K)}(b) - B_{(K+1)}(\{a\})f^{(K)}(a) - \int_a^b f^{(K)}(x)\, dB_{K+1}(\{x\})\right) \\
= \;&\frac{(-1)^{K+1}}{(K+1)!}\left(B_{(K+1)}(\{b\})f^{(K)}(b) - B_{(K+1)}(\{a\})f^{(K)}(a)\right) + \frac{(-1)^K}{(K+1)!}\int_a^b f^{(K)}(x)(K+1)B_K(\{x\})\, dx \\
= \;&\frac{(-1)^{K+1}}{(K+1)!}\left(B_{(K+1)}(\{b\})f^{(K)}(b) - B_{(K+1)}(\{a\})f^{(K)}(a)\right) + \frac{(-1)^K}{K!}\int_a^b f^{(K)}(x)B_K(\{x\})\, dx,
\end{aligned}$$

which is exactly what is to be shown for the induction step. Note, $\frac{d}{dx}B_k(x) = kB_{k-1}(x)$ was used in the third step above.