MATH 155: FINAL EXAMINATION

Due March 11, 2010

For this test, you are free to use your notes, the material available on the course website, and the books that are on reserve at the library. However you may not refer to other books, or use internet resources. You are free to use results proved in class, but you must state clearly the result that you are using. Complete proofs are expected for all questions. There will be partial credit, so please include approaches to the problem, and/or heuristic reasons for what the result might be or why it might be true. Don't get too bogged down by error terms; if you can prove an asymptotic without a very good error term, that still counts for a lot. All the best!

1. Let $\alpha \neq 0$ be a real number, and define $\sigma_{i\alpha}(n) = \sum_{d|n} d^{i\alpha}$. Consider the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{|\sigma_{i\alpha}(n)|^2}{n^s}.$$

In what region does this converge absolutely? Find a meromorphic continuation of $F(s)$ to $\text{Re}(s) > 1/2$ and determine the poles in this region. Find an asymptotic formula for

$$\sum_{n \le x} |\sigma_{i\alpha}(n)|^2.$$

You should give a formula with an error term $O(x^{1-\delta})$ for some $\delta > 0$.

2. Using analytic methods or otherwise, prove that

$$\sum_{n \le x} \frac{d(n)}{\phi(n)} = C_2(\log x)^2 + C_1 \log x + C_0 + O(x^{-\delta})$$

for some constants C_0, C_1 and C_2, and some $\delta > 0$.

3. Let $k \geq 1$ be an integer, and let $\mu_k(n)$ be defined by $(1/\zeta(s))^k = \sum_{n=1}^{\infty} \mu_k(n)/n^s$. Prove that

$$\sum_{n \le x} \mu_k(n) \ll \frac{x}{(\log x)^A},$$

for any $A > 0$.

4. In problem 3 of homework 7 you considered the number of integers n for which $\phi(n) \leq x$, but there you were only asked to guess the asymptotic as the proof

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX was (presumably) difficult. In this problem you'll work out an asymptotic for this. Recall that

$$F(s) = \sum_{n=1}^{\infty} \frac{1}{\phi(n)^s} = \zeta(s)G(s),$$

where

$$G(s) = \prod_p \Big(1 - \frac{1}{p^s} + \frac{1}{(p-1)^s}\Big)$$

is absolutely convergent in $\sigma > 0$.

(i). Show that for $\sigma > 0$, $|(p-1)^{-s} - p^{-s}| \le \min(2(p-1)^{-\sigma}, |s|(p-1)^{-\sigma-1})$ and deduce that if $|t| \ge 3$ and $\sigma > 1 - 1/\log|t|$ then

$$|G(\sigma + it)| \ll (\log|t|)^A,$$

for some constant $A > 0$.

(ii). Prove that

$$N_1(x) = \sum_{\phi(n) \le x} \log \frac{x}{\phi(n)} = G(1)x + O(x\exp(-d\sqrt{\log x})),$$

where $d > 0$ is a constant. Suggestion: Express this as an integral of $F(x)x^s/s^2$ along an appropriate line $\mathrm{Re}(s) = c$. Show that the tails $|t| > T$ of the integral are small. For the line segment $c - iT$ to $c + iT$ move the line of integration to the left, ending up on the line $1 - 1/\log T - iT$ to $1 - 1/\log T + iT$. Choose T carefully.

(iii). Let $N(x)$ denote the number of integers n for which $\phi(n) \le x$. Show that for $1 \le y \le x - 1$,

$$N_1(x+y) - N_1(x) \ge N(x)\log\Big(\frac{x+y}{x}\Big)$$

and

$$\log\Big(\frac{x}{x-y}\Big)N(x) \ge N_1(x) - N_x(x-y),$$

and choosing y carefully deduce an asymptotic formula for $N(x)$.