MATH 155: PROBLEM SET 8

Due May 31

1. For a nice smooth function f with rapid decay show that

$$\sum_{n\in\mathbb{Z}} f(n+\alpha) = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{2\pi ik\alpha}.$$

Prove that for any $x>0$

$$x^{-\frac{1}{2}}\sum_{n\in\mathbb{Z}} e^{-\pi(n+\alpha)^2/x} = \sum_{k\in\mathbb{Z}} e^{-\pi k^2 x+2\pi ik\alpha},$$

and that

$$-2\pi x^{-\frac{3}{2}}\sum_{n\in\mathbb{Z}}(n+\alpha)e^{-\pi(n+\alpha)^2/x} = 2\pi i\sum_{k\in\mathbb{Z}} ke^{-\pi k^2 x+2\pi ik\alpha}.$$

2. Let χ_5 denote the Legendre symbol $\pmod 5$: that is, for all $n\in\mathbb{Z}$ we have $\chi_5(n) = \left(\frac{n}{5}\right)$. This is a Dirichlet character $\pmod 5$, and associated to it is the Dirichlet L-function

$$L(s,\chi_5) = \sum_{n=1}^{\infty}\frac{\chi_5(n)}{n^s}.$$

Let $\theta(t,\chi_5)$ be defined by

$$\theta(t,\chi_5) = \sum_{n\in\mathbb{Z}}\chi_5(n)e^{-\pi n^2 t/5}.$$

Show that for $\mathrm{Re}(s)>1$

$$\left(\frac{5}{\pi}\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi_5) = \frac{1}{2}\int_0^{\infty}\theta(x,\chi_5)x^{s/2}\frac{dx}{x}.$$

3. (a). Show that

$$\theta(x,\chi_5) = \sum_{a=1}^{4}\left(\frac{a}{5}\right)\sum_{n\in\mathbb{Z}} e^{-5\pi x(n+a/5)^2},$$

and invoke the relation for the inner sum from problem 1. Conclude that

$$\theta(x,\chi_5) = \frac{1}{\sqrt{5x}}\sum_{k\in\mathbb{Z}} e^{-\pi k^2/(5x)}\Big(\sum_{a=1}^{4}\left(\frac{a}{5}\right)e^{2\pi ika/5}\Big).$$

(b). If $5|k$ show that the inner sum over a above equals zero. If $5 \nmid k$, using that multiplication by k permutes the reduced residue classes $\pmod 5$, show that the sum over a equals $(\frac{k}{5})\tau(\chi_5)$, where $\tau(\chi_5)$ is the Gauss sum

$$\tau(\chi_5) = \sum_{a=1}^{4} \left(\frac{a}{5}\right) e^{2\pi i a/5}.$$

(c). Conclude that

$$\theta(x, \chi_5) = \frac{\tau(\chi_5)}{\sqrt{5}} \frac{1}{\sqrt{x}} \theta(1/x, \chi_5).$$

(d). Show that $\tau(\chi_5)^2 = 5$, and by computing determine the sign of $\tau(\chi_5)$.

4. Prove the functional equation

$$\left(\frac{5}{\pi}\right)^{s/2} \Gamma(s/2) L(s, \chi_5) = \left(\frac{5}{\pi}\right)^{(1-s)/2} \Gamma((1-s)/2) L(1-s, \chi_5),$$

and show that the above function is holomorphic for all $s \in \mathbb{C}$.