MATH 155: PROBLEM SET 7

Due May 24

1. Revisit the lower bound for $|\zeta(1+it)|$ given in class. Show that for $|t| \geq 1$ we have
$$|\zeta(1+it)| \gg (\log(2+|t|))^{-7}.$$

2. Show that if $|t| \geq 10$ and $\sigma \geq 1 - 1/(\log|t|)$ then
$$|\zeta'(\sigma+it)| \ll (\log|t|)^2.$$

(Note earlier you derived such a bound for $|\zeta'(1+it)|$ and this should follow with small modifications to that result.)

3. Establish, using arguments from class and the above two problems, that for any zero $\beta + i\gamma$ of the Riemann zeta function satisfies
$$\beta \leq 1 - \frac{c}{(\log(2+|\gamma|))^9}.$$

Here $c > 0$ is some constant. Furthermore, show that if $|t| \geq 10$ and $\sigma > 1 - c/(\log|t|)^9$ then $|\zeta(\sigma+it)| \gg (\log|t|)^{-7}$.

4. Incorporate the zero free region above and the corresponding lower bound for zeta into the proof of the prime number theorem given in class. Show how this leads to a quantitative estimate for $\delta(T)$ and $F(T)$ which were left unspecified in that argument. Conclude an asymptotic of the form
$$\psi(x) = x + O(x\exp(-c(\log x)^{\lambda})),$$

for a suitable value of $\lambda > 0$ and some $c > 0$.