MATH 155: PROBLEM SET 3

Due Thursday, April 26.

1. Prove that the 3-divisor function $d_3(n)$ is at most $C(\epsilon)n^\epsilon$ for any $\epsilon > 0$ and some positive constant $C(\epsilon)$. Determine an explicit constant C such that $d_3(n) \le Cn^{1/4}$ for all n.

2. Show that
$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu(d)^2}{\phi(d)}.$$
Show that
$$\sum_{n\le x} \frac{n}{\phi(n)} = cx + O(\log x)$$
for some constant c. Show that $c = \zeta(2)\zeta(3)/\zeta(6)$.

3. (a). Show that
$$\sum_{pq\le x} \frac{1}{pq} = \Big(\sum_{p\le x}\frac{1}{p}\Big)^2 - 2\Big(\sum_{p\le\sqrt{x}}\frac{1}{p}\Big)\Big(\sum_{x/p<q\le x}\frac{1}{q}\Big) - \Big(\sum_{\sqrt{x}<p\le x}\frac{1}{p}\Big)^2.$$

(b). For $p \le \sqrt{x}$ show that
$$\sum_{x/p<q\le x}\frac{1}{q} = \log\log x - \log\log(x/p) + O\Big(\frac{1}{\log x}\Big) = O\Big(\frac{\log p}{\log x}\Big).$$

(c). Conclude that
$$\sum_{pq\le x}\frac{1}{pq} = \Big(\sum_{p\le x}\frac{1}{p}\Big)^2 + O(1).$$

(d). Let B be the constant in the asymptotic
$$\sum_{p\le x}\frac{1}{p} = \log\log x + B + O\Big(\frac{1}{\log x}\Big).$$
Show that
$$\sum_{n\le x}(\omega(n) - \log\log x - B)^2 \sim x\log\log x.$$