MATH 155: PROBLEM SET 1

Due Thursday, April 12.

1. Find asymptotic formulae for $\sum_{n\le x} \frac{\log n}{n}$ and $\sum_{n\le x} n^{-1/\pi}$. Your error terms should be of size $O((\log x)/x)$ and $O(x^{-1/\pi})$ respectively.

2. Give an example of a non-negative function f such that $f(x) = o(x^\epsilon)$ for any fixed $\epsilon > 0$, but such that $(\log x)^A = o(f(x))$ for any fixed $A > 0$. We would say colloquially that f grows faster than any power of $\log x$ but slower than any power of x.

3. For all large n prove that $\omega(n) \ll \log n/\log\log n$ where $\omega(n)$ counts the number of distinct prime factors of n.

4. In class we showed that

$$N! = C\sqrt{N}\Big(\frac{N}{e}\Big)^N\Big(1+O\Big(\frac{1}{N}\Big)\Big)$$

for some positive constant C. In this (challenging) exercise we shall determine that $C = \sqrt{2\pi}$.

(a). In the formula $2^{2N} = \sum_{k=0}^{2N}\binom{2N}{k}$ show that the terms are increasing in the range $0 \le k \le N$ and decreasing thereafter.

(b). If $k = N + \ell$ with $|\ell| \le N^{2/3}$ show that

$$\binom{2N}{k} = \frac{\sqrt{2}}{C\sqrt{N}} 2^{2N} e^{-\ell^2/N + O(|\ell|^3/N^2)}.$$

You may find useful here to recall the Taylor approximation for $\log(1+x)$ for small values of x.

(c). Using (a) and (b) explain why for large N we must have

$$2^{2N} \sim \frac{\sqrt{2}}{C\sqrt{N}} 2^{2N} \int_{-\infty}^{\infty} e^{-t^2/N}\, dt,$$

and conclude that $C = \sqrt{2\pi}$.

5. This exercise recaps and makes precise the Euler-Maclaurin summation formula that we discussed in class. The Bernoulli polynomials $B_k(x)$ for $k \ge 0$ are defined by setting $B_0(x) = 1$ for all x, and for $k \ge 1$ we have

$$\frac{d}{dx}B_k(x) = kB_{k-1}(x)$$

and

$$\int_0^1 B_k(x)dx = 0.$$

You may check that $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$ etc.

Using induction on K establish the Euler-Maclaurin formula

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)dx + \sum_{k=1}^{K} \frac{(-1)^k}{k!}(B_k(\{b\})f^{(k-1)}(b) - B_k(\{a\})f^{(k-1)}(a))$$
$$- \frac{(-1)^K}{K!}\int_a^b B_K(\{x\})f^{(K)}(x)dx.$$