DIRICHLET'S THEOREM ON PRIMES IN PROGRESSIONS, II

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In the previous article we established Dirichlet's theorem when the modulus is 4. Let us now consider a few more cases of that argument until we can see clearly the strategy of the general proof.

Let us begin with the case $q = 3$. In analogy with our argument for $q = 4$ we introduce the function

$$\chi_{-3}(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 3 \\ 0 & \text{if } 3|n \\ -1 & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

Notice that $\chi_{-3}(n)$, like χ_{-4}, is a multiplicative function. Define

$$L(s, \chi_{-3}) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \dots ,$$

and note that the series converges absolutely when $s > 1$, and conditionally (using the alternating series test) when $s > 0$. Moreover by the multiplicativity of χ_{-3} we see that for $s > 1$ there holds the Euler product

$$L(s, \chi_{-3}) = \prod_p \Big(1 - \frac{\chi_{-3}(p)}{p^s}\Big)^{-1}. \tag{1}$$

Taking logarithms in (1) we find that for $s > 1$

$$\begin{aligned} \log L(s, \chi_{-3}) &= \sum_p -\log\Big(1 - \frac{\chi_{-3}(p)}{p^s}\Big) = \sum_p \Big(\frac{\chi_{-3}(p)}{p^s} + O\Big(\frac{1}{p^{2s}}\Big)\Big) \\ &= \sum_p \frac{\chi_{-3}(p)}{p^s} + O(1). \end{aligned}$$

Recalling that

$$\log \zeta(s) = \sum_p \frac{1}{p^s} + O(1),$$

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we obtain that

$$2\sum_{\substack{p\\ p\equiv 1 \pmod 3}} \frac{1}{p^s} = \log\zeta(s) + \log L(s,\chi_{-3}) + O(1), \tag{2}$$

and that

$$2\sum_{\substack{p\\ p\equiv 2 \pmod 3}} \frac{1}{p^s} = \log\zeta(s) - \log L(s,\chi_{-3}) + O(1). \tag{3}$$

As in the case $\pmod 4$ we see that the proof of the infinitude of primes in the residue classes $\pmod 3$ rests on the behavior of $L(s,\chi_{-3})$ as $s \to 1^+$. If this limiting value is not zero or infinity then we are done! Now, as $s\to 1^+$ we see that

$$L(s,\chi_{-3}) \to L(1,\chi_{-3}) = \frac{1}{1} - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \dots ,$$

and this sum converges (and so is not infinity), and moreover is visibly positive (and so is not zero). Moreover, as in the case of χ_{-4} we may even evaluate $L(1,\chi_{-3})$. Namely,

$$\begin{aligned} L(1,\chi_{-3}) &= \int_0^1 (1-t+t^3-t^4+t^6-t^7+\dots)dt = \int_0^1 (1-t)(1+t^3+t^6+\dots)dt \\ &= \int_0^1 \frac{1-t}{1-t^3}dt = \int_0^1 \frac{dt}{t^2+t+1}. \end{aligned}$$

Using this in (2) and (3) we conclude:

Corollary 1. *As $s\to 1^+$ we have*

$$\sum_{p\equiv 1 \pmod 3} \frac{1}{p^s} = \frac{1}{2}\log\frac{1}{s-1} + O(1),$$

and

$$\sum_{p\equiv 2 \pmod 3} \frac{1}{p^s} = \frac{1}{2}\log\frac{1}{s-1} + O(1).$$

In particular, there are infinitely many primes in the residue classes $1 \pmod 3$ *and* $2 \pmod 3$.

Next let us turn to the case $q = 8$. Here we must distinguish between four congruence classes $1 \pmod 8$, $3 \pmod 8$, $5 \pmod 8$ and $7 \pmod 8$. Accordingly we look for four functions which are multiplicative, and which are periodic with period 8, and we shall use these to construct more L-functions. We have already encountered two of these: namely, the function 1 on all natural numbers which is periodic with period 8 and multiplicative and which gives rise to $\zeta(s)$, and the function χ_{-4} defined in our previous notes which is periodic with period 4 and therefore with period 8 and which gave rise to $L(s, \chi_{-4})$. There are two further functions: these are

$$\chi_8(n) = \begin{cases} 1 & \text{if } n \equiv 1, 7 \pmod 8 \\ 0 & \text{if } 2|n \\ -1 & \text{if } n \equiv 3, 5 \pmod 8), \end{cases}$$

and

$$\chi_{-8}(n) = \begin{cases} 1 & \text{if } n \equiv 1, 3 \pmod 8 \\ 0 & \text{if } 2|n \\ -1 & \text{if } n \equiv 5, 7 \pmod 8). \end{cases}$$

From their definitions it is evident that these are periodic with period 8 and you should verify that they are also multiplicative. You may also note the interesting fact that $\chi_8(n)\chi_{-4}(n) = \chi_{-8}(n)$. Corresponding to χ_8 and χ_{-8} we define the L-functions

$$L(s, \chi_8) = \sum_{n=1}^{\infty} \frac{\chi_8(n)}{n^s} = \prod_p \Big(1 - \frac{\chi_8(p)}{p^s}\Big)^{-1},$$

and

$$L(s, \chi_{-8}) = \sum_{n=1}^{\infty} \frac{\chi_{-8}(n)}{n^s} = \prod_p \Big(1 - \frac{\chi_{-8}(n)}{n^s}\Big)^{-1}.$$

Taking logarithms we obtain as before that

$$\log L(s, \chi_8) = \sum_p \frac{\chi_8(p)}{p^s} + O(1),$$

and

$$\log L(s, \chi_{-8}) = \sum_p \frac{\chi_{-8}(p)}{p^s} + O(1).$$

We may use χ_4, χ_8, χ_{-8} (together with the function that is 1 always) to distinguish between the four reduced residue classes $\pmod 8$. If n is odd note that

$$\frac{1}{4}\Big(1 + \chi_{-4}(n) + \chi_8(n) + \chi_{-8}(n)\Big) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 8 \\ 0 & \text{if not.} \end{cases}$$

Similarly

$$\frac{1}{4}\Big(1 - \chi_{-4}(n) - \chi_8(n) + \chi_{-8}(n)\Big) = \begin{cases} 1 & \text{if } n \equiv 3 \pmod 8 \\ 0 & \text{otherwise;} \end{cases}$$

$$\frac{1}{4}\Big(1 + \chi_{-4}(n) - \chi_8(n) - \chi_{-8}(n)\Big) = \begin{cases} & \text{if } n \equiv 5 \pmod 8 \\ 0 & \text{otherwise;} \end{cases}$$

and lastly

$$\frac{1}{4}\Big(1-\chi_{-4}(n)+\chi_8(n)-\chi_{-8}(n)\Big)=\begin{cases}1 & \text{if } n\equiv 7 \pmod 8\\ 0 & \text{otherwise.}\end{cases}$$

It therefore follows that (for $s>1$)

$$\sum_{\substack{p\\ p\equiv 1 \pmod 8}}\frac{1}{p^s}=\frac{1}{4}\Big(\log\zeta(s)+\log L(s,\chi_{-4})+\log L(s,\chi_8)+\log L(s,\chi_{-8})\Big)+O(1),$$

and similar expressions with varying signs hold for the sums over primes in the other three residue classes. We know that as $s\to 1^+$ the $\log\zeta(s)$ term becomes $\log(1/(s-1))+O(1)$, and to obtain the infinitude of primes in the progressions $\pmod 8$ we must show that the other three L-functions tend neither to zero nor infinity as $s\to 1^+$. We have already seen this for $L(s,\chi_{-4})$ which tends to $\pi/4$ as $s\to 1^+$. As $s\to 1^+$ we see that

$$L(s,\chi_8)\to 1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}-\frac{1}{11}-\frac{1}{13}+\frac{1}{15}+\dots,$$

which by grouping every four terms together we may see both converges, and is strictly positive. Moreover in your homework you will even have determined its value. Similar conclusions apply to $L(s,\chi_{-8})$ and we finally deduce

Corollary 2. *As $s\to 1^+$ and for $a=1$, 3, 5, or 7 we have*

$$\sum_{p\equiv a \pmod 8}\frac{1}{p^s}=\frac{1}{4}\log\frac{1}{s-1}+O(1),$$

and so there are infinitely many primes $\equiv a \pmod 8$.

Now let us consider the case $q=5$. We must find primes in the four progressions 1, 2, 3 or 4 $\pmod 5$, and in order to distinguish these four cases we seek four multiplicative functions that are also periodic $\pmod 5$. As usual one of these is the function that is one always. Another such function is the Legendre symbol $\pmod 5$; we call this χ_5 so that

$$\chi_5(n)=\begin{cases}1 & \text{if } n\equiv 1,4 \pmod 5\\ 0 & \text{if } 5|n\\ -1 & \text{if } n\equiv 2,3 \pmod 5.\end{cases}$$

How should we find mulitplicative functions that are periodic $\pmod 5$? Let χ denote such a function, and we suppose that χ is not identically zero; note that this means $\chi(n)=\chi(1\cdot n)=\chi(1)\chi(n)$ so that $\chi(1)=1$. We also suppose that $\chi(n)=0$ if $5|n$ since we are only interested in the reduced residue classes $\pmod 5$. Since χ is periodic $\pmod 5$, $\chi(n)$ depends only on $n \pmod 5$. Note that the group of reduced residues $\pmod 5$ is generated by the residue class $2 \pmod 5$. Thus if we define $\chi(2)$ then $\chi(4)=\chi(2)^2$, $\chi(2)^3=\chi(2^3)=\chi(3)$, and $\chi(2)^4=\chi(2^4)=\chi(1)$ and χ would be defined on all integers. Our last observation also gives that $\chi(2)^4=\chi(1)=1$ so that the only way to define $\chi(2)$ is to take it to be a fourth root of unity: namely $\chi(2) = 1$, -1, i or $-i$. Each of these choices for $\chi(2)$ gives rise to a periodic $\pmod 5$ multiplicative function. Taking $\chi(2) = 1$ gives essentially the function that is one always although we see now that it is more consistent to set it to be 1 if $(n, 5) = 1$ and 0 otherwise; we call this function χ_0. Taking $\chi(2) = -1$ gives us the Legendre symbol χ_5 mentioned above. We shall denote the choice i by just χ; so χ is the function that is given by $\chi(2) = i$, $\chi(3) = -i$ and $\chi(4) = -1$. The other choice of $-i$ gives the complex conjugate of χ; namely $\overline{\chi}(2) = -i$, $\overline{\chi}(3) = i$, $\overline{\chi}(4) = -1$ etc. We may summarize our findings in the following table:

n	1	2	3	4
χ_0	1	1	1	1
χ_5	1	-1	-1	1
χ	1	i	$-i$	-1
$\overline{\chi}$	1	$-i$	i	$-1.$

To each of these four functions χ_0, χ_5, χ and $\overline{\chi}$ we may associate L-functions which are given by a series over all natural numbers and also by a product over primes. Note that

$$L(s, \chi_0) = \prod_p \Big(1 - \frac{\chi_0(p)}{p^s}\Big)^{-1} = \prod_{p \nmid q} \Big(1 - \frac{1}{p^s}\Big)^{-1} = \zeta(s) \prod_{p | q} \Big(1 - \frac{1}{p^s}\Big),$$

so that for $s > 1$

(4) $$\log L(s, \chi_0) = \log \zeta(s) + \sum_{p|q} \log(1 - 1/p^s) = \log \zeta(s) + O(1) = \log \frac{1}{(s-1)} + O(1).$$

Next,

(5) $$\log L(s, \chi_5) = \sum_p \frac{\chi_5(p)}{p^s} + O(1).$$

The expressions for $\log L(s, \chi)$ and $\log L(s, \overline{\chi})$ are similar, except that we are now taking logarithms of complex numbers. All that we use is that if z is a complex number with $|z| < 1$ then, as in the real case, we may write a series expansion $\log(1+z) = z - z^2/2 + z^3/3 - \dots$ which will converge and have the property that its exponential will give $1+z$. The complex log is multivalued as we can add any multiple of $2\pi i$ and use that $e^{2\pi i} = 1$, but the series that we chose is the natural one, because as $z \to 0$ one would like to have $\log(1 + z) \to 0$ as well. In any case, we have for $|z| < 1$, $\log(1 + z) = z + O(z^2)$ as before, and hence

(6) $$\log L(s, \chi) = \sum_p \frac{\chi(p)}{p^s} + O(1),$$

and

(7) $$\log L(s, \overline{\chi}) = \sum_p \frac{\overline{\chi}(p)}{p^s} + O(1).$$

By inspecting our table above we may write each of the four reduced residue classes $\pmod 5$ as a linear combination of our functions χ_0, χ_5, χ and $\overline{\chi}$. For example, if $5 \nmid n$ then $\frac{1}{4}(\chi_0(n) + \chi_5(n) + \chi(n) + \overline{\chi}(n))$ is 1 if $n \equiv 1 \pmod 5$ and 0 otherwise. Similarly $\frac{1}{4}(\chi_0(n) - \chi_5(n) - i\chi(n) + i\overline{\chi}(n))$ is 1 if $n \equiv 2 \pmod 5$ and 0 otherwise. You should find the analogous expressions for $n \equiv 3 \pmod 5$ and $n \equiv 4 \pmod 5$. Using these expressions in (4, 5, 6, 7) we find that

$$\sum_{p \equiv 1 \pmod 5} \frac{1}{p^s} = \frac{1}{4}\Big(\log L(s,\chi_0) + \log L(s,\chi_5) + \log L(s,\chi) + \log L(s,\overline{\chi})\Big) + O(1),$$

and similar expressions hold for the other residue classes; for example,

$$\sum_{p \equiv 2 \pmod 5} \frac{1}{p^s} = \frac{1}{4}\Big(\log L(s,\chi_0) - \log L(s,\chi_5) - i\log L(s,\chi) + i\log L(s,\overline{\chi})\Big) + O(1).$$

By (4) we see that the $\log L(s,\chi_0)$ term gives a contribution that goes to infinity as $s \to 1^+$. So all will be well if we can show once again that the other L-functions go to values not equal to zero or infinity as $s \to 1^+$.

Now

$$L(s,\chi_5) = \Big(\frac{1}{1^s} - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s}\Big) + \Big(\frac{1}{6^s} - \frac{1}{7^s} - \frac{1}{8^s} + \frac{1}{9^s}\Big) + \dots$$

and grouping terms as above we may see both that the series converges to some finite answer as $s \to 1^+$, and also that the answer is strictly positive. Moreover in your homework you were asked to evaluate this. Similarly

$$L(s,\chi) = \Big(1 + \frac{i}{2^s} - \frac{i}{3^s} - \frac{1}{4^s}\Big) + \dots,$$

and we see that the real and imaginary parts give alternating series that converge; moreover the real parts and imaginary parts above are seen to be strictly positive. Similarly for $L(s,\overline{\chi})$ which is just the complex conjugate of $L(s,\chi)$ (so that its real part is positive, and its imaginary part negative). We have thus proved Dirichlet's theorem $\pmod 5$!

Corollary 3. *As $s \to 1^+$ and for $a = 1$, 2, 3, or 4 we have*

$$\sum_{p \equiv a \pmod 5} \frac{1}{p^s} = \frac{1}{4}\log\frac{1}{(s-1)} + O(1),$$

and in particular there are infinitely many primes $p \equiv a \pmod 5$.