MATH 152 Problem set 9 solutions

1. In both (a) and (b), we simply expand the left-hand side and use orthogonality relations. For (a),

$$\begin{aligned}
&\sum_{\chi \pmod q} \Big|\sum_{n=1}^{q} c_n\chi(n)\Big|^2 \\
&= \sum_{\chi}\Big(\sum_{n=1}^{q} c_n\chi(n)\Big)\Big(\sum_{m=1}^{q} \overline{c}_m\overline{\chi}(m)\Big) \\
&= \sum_{\chi}\sum_{1\le n,m\le q} c_n\overline{c}_m\chi(n)\overline{\chi}(m) \\
&= \sum_{\chi}\sum_{n} c_n\overline{c}_n\chi(n)\overline{\chi}(n)
\end{aligned}$$

by the second orthogonality relation. Since $\chi(n)\overline{\chi}(n) = 1$ if $(n,q) = 1$ and 0 otherwise, this equals

$$\sum_{\chi}\sum_{n,(n,q)=1} |c_n|^2 = \phi(q)\sum_{n,(n,q)=1} |c_n|^2.$$

Similarly, for (b),

$$\begin{aligned}
&\sum_{n=1}^{q}\Big|\sum_{\chi \pmod q} c_\chi\chi(n)\Big|^2 \\
&= \sum_{n=1}^{q}\Big(\sum_{\chi \pmod q} c_\chi\chi(n)\Big)\Big(\sum_{\psi \pmod q} \overline{c}_\psi\overline{\psi}(n)\Big) \\
&= \sum_{n}\sum_{\chi,\psi} c_\chi\overline{c}_\psi\chi(n)\overline{\psi}(n) \\
&= \sum_{n}\sum_{\chi} c_\chi\overline{c}_\chi\chi(n)\overline{\chi}(n) \\
&= \sum_{n,(n,q)=1}\sum_{\chi} |c_\chi|^2,
\end{aligned}$$

where the second last line follows from the first orthogonality.

2. Note that

$$d(n)\chi(n) = \sum_{ab=n}\chi(a)\chi(b)$$

so we will to estimate

$$\sum_{n\leq x}\sum_{ab=n}\chi(a)\chi(b)=\sum_{ab\leq x}\chi(a)\chi(b).$$

Using Dirichlet's hyperbola method, this equals

$$\sum_{a\leq\sqrt{x}}\sum_{b\leq x/a}\chi(a)\chi(b)+\sum_{b\leq\sqrt{x}}\sum_{\sqrt{x}<a\leq x/b}\chi(a)\chi(b).$$

The magnitude of the first sum is bounded by

$$\begin{aligned}
&\Big|\sum_{a\leq\sqrt{x}}\chi(a)\sum_{b\leq x/a}\chi(b)\Big|\\
\leq\;&\sum_{a\leq\sqrt{x}}|\chi(a)|\Big|\sum_{b\leq x/a}\chi(b)\Big|\\
\leq\;&\sum_{a\leq\sqrt{x}}|\chi(a)|O(1)\qquad\text{(because }\chi\text{ is nonprincipal)}\\
\leq\;&\sqrt{x}O(1).
\end{aligned}$$

Similarly, the second sum is bounded by $\sqrt{x}O(1)$. Therefore

$$\sum_{ab\leq x}\chi(a)\chi(b)=O(\sqrt{x}).$$

3. (i) follows upon observing that

$$\frac{\zeta(4)}{\zeta(2)}=\prod_{q\text{ prime}}\frac{1-q^{-2}}{1-q^{-4}}=\prod_{q\text{ prime}}(1+q^{-2})^{-1}$$

$$L(2,\chi)=\prod_{q\text{ prime}}(1-\Big(\frac{q}{p}\Big)q^{2})^{-1},\ \zeta(2)=\prod_{q\text{ prime}}(1-q^{2})^{-1}$$

and that

$$(1+q^{-2})^{-1}\leq(1-\Big(\frac{q}{p}\Big)q^{2})^{-1}\leq(1-q^{2})^{-1}.$$

(ii) Recall

$$L(2,\chi)=\sum_{n=1}^{\infty}\Big(\frac{n}{p}\Big)n^{-2},\ \zeta(2)=\sum_{n=1}^{\infty}n^{-2}.$$

In order to make $|L(2,\chi)-\zeta(2)|$ arbitrarily small, we must pick a prime such that $(\frac{n}{p})=1$ for all $1\le n\le N$, where N is an arbitrarily large number. By Dirichlet's theorem, there exist infinitely many primes p that is 1 mod 8, 1 mod p_1, 1 mod p_2, ..., 1 mod p_k, where $\{p_i\}$ is an enumeration of odd primes in increasing order and p_k is the smallest prime greater than N. By construction $(\frac{2}{p})=1$, $(\frac{p_i}{p})=(\frac{p}{p_i})=1$ for all $i=1,\dots,k$ by quadratic reciprocity, and so $(\frac{n}{p})=1$ for all $1\le n\le N$. Therefore

$$|L(2,\chi)-\zeta(2)|<2\sum_{n>N}\frac{1}{n^2}<\frac{1}{N}.$$

This solves the first part of the problem.

For the second part, we use the same idea again, but the execution is a little bit trickier. By Exercise 4 in Problem Set 6, we have

$$\begin{aligned}\frac{\zeta(4)}{\zeta(2)} &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}\sum_{m=1}^{\infty}\frac{1}{m^4}\\ &= \sum_{m,n\ge 2}\frac{\mu(n)}{(m^2n)^2}\\ &= \sum_{\substack{m\ge 1\\ n \text{ square free}}}\frac{\mu(n)}{(m^2n)^2}.\end{aligned}$$

Since every positive integer can be factorized uniquely into a square and a squarefree integer, we can rewrite this so that

$$\frac{\zeta(4)}{\zeta(2)}=\sum_{n=1}^{\infty}\frac{\nu(n)}{n^2}$$

where $\nu(n):=\mu(\text{squarefree part of } n)$.

We want to show that there exist infinitely many primes p such that

$$\left|L(2,\chi)-\frac{\zeta(4)}{\zeta(2)}\right|=\Big|\sum_{n=1}^{\infty}\Big(\frac{n}{p}\Big)\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{\nu(n)}{n^2}\Big|$$

is small. It suffices to show that we can pick p so that $(\frac{n}{p})=\nu(n)$ for all $1\le n\le N$, where N is arbitrarily large.

As before let $\{p_i\}$ be an enumeration of odd primes in increasing order, and let p_k be the smallest prime greater than N. For each $1\le i\le k$ pick a nonresidue $a_i \pmod{p_i}$. Dirichlet's theorem implies that there are infinitely many primes p that is 5 mod 8 and a_i mod p_i for all $1 \le i \le k$. Then $(\frac{2}{p}) = -1$, and $(\frac{p_i}{p}) = (\frac{p}{p_i}) = -1$ by quadratic reciprocity. Therefore, if $1 \le n \le N$ and $n = a^2 b$ where b is the squarefree part of n, then $(\frac{n}{p}) = (\frac{b}{p}) = \mu(b) = \nu(n)$, as desired.

4. (a) By Euclidean algorithm we can write $x = qK + r$ with $0 \le r < q$. Now

$$\begin{aligned}\sum_{n \le x, (n,q)=1} 1 &= K\phi(q) + O(\phi(q)) \\ &= \frac{x-r}{q}\phi(q) + O(\phi(q)) \\ &= \frac{\phi(q)}{q}x + O(\phi(q)).\end{aligned}$$

(b) We will show that $\mathcal{B}^c = \{n : 2010 \nmid \phi(n)\}$ has density 0. $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, so we can rewrite

$$\mathcal{B}^c = \{n : 2 \nmid \phi(n)\} \cup \{n : 3 \nmid \phi(n)\} \cup \{n : 5 \nmid \phi(n)\} \cup \{n : 67 \nmid \phi(n)\}.$$

Hence it suffices to prove a more general claim that $\{n : p \nmid \phi(n)\}$ has density 0 for every prime p. When $p = 2$, from the formula

$$\phi(p_1^{a_1} \dots p_k^{a_k}) = (p_1 - 1)p_1^{a_1 - 1} \dots (p_k - 1)p_k^{a_k - 1}$$

it is clear that $p \nmid \phi(n)$ unless $n = 2$. So we may assume p is odd.

$\{n : p \nmid \phi(n)\}$ is contained in the set $A := \{n : n \text{ has no prime factor } 1 \bmod p\}$. Let's show that A has density 0. Enumerate 1 (mod p) primes in increasing order as $\{p_i\}$. Also choose some positive integer k. Note that $A \subset A_k := \{n : n \text{ is coprime to } p_1, \dots, p_k\}$. So we have reduced the problem to showing that the density of A_k approaches zero as $k \to \infty$.

Let's estimate the density of A_k. Pick some $x > k$. The number of $n \le x$ such that n is coprime with $p_1, \dots, p_k$ equals

$$\sum_{\substack{n \le x \\ (n, p_1 \dots p_k) = 1}} 1 = \frac{(p_1 - 1) \dots (p_k - 1)}{p_1 \dots p_k} x + O(\phi(p_1 \dots p_k)),$$

where the estimate follows from (a). Dividing this by x, this equals

$$\prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) + \frac{1}{x} O(\phi(p_1 \dots p_k)).$$

Letting $x \to \infty$, this approaches $\prod_{i=1}^{k}(1 - p_i^{-1})$. Hence (density of A_k) $= \prod_{i=1}^{k}(1 - p_i^{-1})$.

Furthermore, $\prod_{i=1}^{k}(1-p_i^{-1})$ converges to zero as $k \to \infty$, because $\prod_{i=1}^{\infty}(1-p_i^{-1})^{-1}$ diverges to infinity (because $\log \prod_{i=1}^{\infty}(1 - p_i^{-1})^{-1} = \sum_i - \log(1 - p_i^{-1}) = \sum_i 1/p_i$+convergent, and $\sum_i 1/p_i$ diverges by Dirichlet). This completes the proof.