MATH 152 Problem set 8 solutions

1. Let $a_n = 1$ (for all n), $f(n) = \log n$, $S(n) = \sum_{k=1}^{n} a_k$ and apply the partial summation formula (as in the November 8th notes) to obtain

$$\sum_{n=1}^{N} \log n = N \log N - \sum_{n=1}^{N-1} n\Big(\log(n+1) - \log n\Big).$$

It remains to rewrite the sum on the right-hand side in terms of integrals. First of all, note

$$\log(n+1) - \log n = \int_n^{n+1} \frac{1}{t} dt,$$

so

$$\begin{aligned} n\Big(\log(n+1) - \log n\Big) &= \int_n^{n+1} \frac{n}{t} dt \\ &= \int_n^{n+1} \frac{[t]}{t} dt \\ &= \int_n^{n+1} 1 + \frac{-\{t\}}{t} dt \\ &= 1 - \int_n^{n+1} \frac{-\{t\}}{t} dt. \end{aligned}$$

This implies

$$\sum_{n=1}^{N-1} n\Big(\log(n+1) - \log n\Big) = \sum_{n=1}^{N-1} \Big(1 - \int_n^{n+1} \frac{\{t\}}{t} dt\Big) = N - 1 - \int_1^N \frac{\{t\}}{t} dt.$$

Plugging this result into the earlier equality above proves $\sum_{n=1}^{N} \log n = N \log N - N + 1 + \int_1^N \frac{\{t\}}{t} dt$.

Next let's think about $F(x) := \int_1^x \{t\} dt$. Notice that we can write

$$F(x) = \frac{1}{2}([x] - 1) + \frac{1}{2}\{x\}^2$$

or, equivalently

$$F(x) = \frac{1}{2}(x - \{x\} - 1) + \frac{1}{2}\{x\}^2 = \frac{1}{2}x + \frac{1}{2}(\{x\}^2 - \{x\} - 1).$$

$F(x)$ is an antiderivative of $\{x\}$ in $[1, \infty)$ except at when x is an integer. Since we can ignore the set of integers when doing an integral, we can apply integration by parts and write

$$\int_1^N \frac{\{t\}}{t} dt = \frac{F(N)}{N} - \frac{F(1)}{1} - \int_1^N \frac{F(t)}{-t^2} dt = \frac{F(N)}{N} + \int_1^N \frac{F(t)}{t^2} dt.$$

The first term on the right is clearly $1/2 + O(1/N)$. As for the integral, first notice that

$$\int_1^\infty \frac{\{x\}^2 - \{x\} - 1}{2t^2} dt$$

is a convergent integral because the integrand is bounded by 2, and that

$$-\frac{1}{N} = \int_N^\infty \frac{-2}{2t^2} dt < \int_N^\infty \frac{\{x\}^2 - \{x\} - 1}{2t^2} dt < \int_N^\infty \frac{2}{2t^2} dt = \frac{1}{N}.$$

So

$$\begin{aligned}
&\int_1^N \frac{F(t)}{t^2} dt \\
&= \int_1^N \frac{1}{2t} dt + \int_1^N \frac{\{x\}^2 - \{x\} - 1}{2t^2} dt \\
&= \frac{1}{2}\log N + \int_1^\infty \frac{\{x\}^2 - \{x\} - 1}{2t^2} dt - \int_N^\infty \frac{\{x\}^2 - \{x\} - 1}{2t^2} dt \\
&= \frac{1}{2}\log N + \int_1^\infty \frac{\{x\}^2 - \{x\} - 1}{2t^2} dt + O(\frac{1}{N}).
\end{aligned}$$

Therefore

$$\int_1^N \frac{\{t\}}{t} dt = \frac{1}{2}\log N + C_0 + O(1/N)$$

where $C_0 = 1/2 + \int_1^\infty (\{x\}^2 - \{x\} - 1)/2t^2 dt$, as desired.

Finally, combining our results so far we get

$$\log N! = \sum_{n=1}^N \log n = N\log N - N + 1 + \frac{1}{2}\log N + C_0 + O(\frac{1}{N}),$$

which quickly implies

$$N! = e^{N\log N} e^{-N+1} e^{\frac{1}{2}\log N} e^{C_0 + O(\frac{1}{N})}.$$

By rearranging the terms a little, this becomes

$$N^N e^{-N}\sqrt{N} e^{C_0+1+O(\frac{1}{N})}.$$

Hence

$$\frac{N!}{N^N e^{-N}\sqrt{N} e^{C_0+1}} = e^{O(\frac{1}{N})} \longrightarrow 1$$

as $N \to \infty$.

2. The "obvious estimation" of the tail $\sum_{n>N} 1/n^2$: simply observe that

$$0 < \sum_{n>N} \frac{1}{n^2} < \int_N^\infty \frac{1}{x^2} dx = \frac{1}{N}.$$

This is an estimate on $\sum_{n>N} 1/n^2$ with the margin of error = RHS - LHS= $1/N$.

The "refined estimation": we can improve the lower bound in the previous one, like this:

$$\frac{1}{N+1} = \int_{N+1}^\infty \frac{1}{x^2} dx < \sum_{n=N+1}^\infty \frac{1}{n^2} < \int_N^\infty \frac{1}{x^2} dx = \frac{1}{N}.$$

This is an estimate with the margin of error $1/N - 1/(N+1) = 1/N(N+1) \sim 1/N^2$.

This much is all you are asked to do, but it is possible (and not hard at all) to give a still better estimate, with the margin of error about $O(1/N^3)$. Above we approximated each term $1/n^2$ by $\int_n^{n+1} 1/x^2 dt$. For the better estimate, we approximate $1/n^2$ by $\int_n^{n+1} 1/x^2 dt + \frac{1}{2}(1/n^2 - 1/(n+1)^2)$. (Draw the graph of $y = 1/x^2$ and compare the area beneath the curve with the graph of $y = 1/[x]^2$ to understand why this has a possibility to give a better estimate). The error of this individual term estimate equals (again, drawing the graphs will help understand where the following integral comes from)

$$\begin{aligned}
&\int_n^{n+1} \Big(\frac{1}{(n+1)^2} - \frac{1}{n^2}\Big)(x-n) + \frac{1}{n^2} - \frac{1}{x^2} dx \\
= \quad & \Big(\frac{1}{(n+1)^2} - \frac{1}{n^2}\Big)(\frac{1}{2}x^2 - nx) - \frac{x}{n^2} + \frac{1}{x}\Big|_n^{n+1} \\
= \quad & \Big(\frac{1}{(n+1)^2} - \frac{1}{n^2}\Big)\frac{1}{2} + \frac{1}{n^2} - \frac{1}{(n+1)n} \\
= \quad & \frac{-n-1/2}{(n+1)^2 n^2} + \frac{1}{n^2(n+1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1/2}{(n+1)^2 n^2} \\
&= O(\frac{1}{n^4}).
\end{aligned}
$$

Hence the sum of all the individual error terms is $\sum_{n>N} O(\frac{1}{n^4}) = O(\frac{1}{n^3})$.

3. Assume $s > 1$. Put $a_n = 1$ and $f(n) = 1/n^s$ and apply the partial summation formula to get

$$\sum_{n=1}^{N} \frac{1}{n^s} = \frac{N}{N^s} - \sum_{n=1}^{N} n\Big(\frac{1}{(n+1)^s} - \frac{1}{n^s}\Big).$$

As $N \to \infty$, the left-hand side converges to $\zeta(s)$, and N/N^s on the right vanishes. To rewrite the sum on the right as an integral, we basically repeat what we did in Problem 1 above. First of all, note

$$
\begin{aligned}
n\Big(\frac{1}{(n+1)^s} - \frac{1}{n^s}\Big) &= -s \int_n^{n+1} \frac{n}{t^{s+1}} dt \\
&= -s \int_n^{n+1} \frac{[t]}{t^{s+1}} dt \\
&= -s \int_n^{n+1} \frac{1}{t^s} dt + s \int_n^{n+1} \frac{\{t\}}{t^{s+1}} dt.
\end{aligned}
$$

Hence

$$\sum_{n=1}^{\infty} n\Big(\frac{1}{(n+1)^s} - \frac{1}{n^s}\Big) = -s \int_1^{\infty} \frac{1}{t^s} dt + s \int_1^{\infty} \frac{\{t\}}{t^{s+1}} dt = \frac{-s}{s-1} + s \int_1^{\infty} \frac{\{t\}}{t^{s+1}} dt.$$

Substitute this into the earlier equality to obtain the desired result.

Next let's consider the integral $\int_1^{\infty} \frac{\{t\}}{t^{s+1}} dt$. This is bounded from below by 0 and from above by $\int_1^{\infty} \frac{1}{t^{s+1}} dt$, which exists for all $s > 0$. So our integral converges for all $s > 0$. In case $s = 0$, however, we have already shown in Problem 1 that our integral will grow like $\frac{1}{2} \log N$. And of course, when $s < 0$ it can only be worse.

4. Write $S(n) = \sum_{k=1}^{n} a_n$. By assumption $\lim_{n\to\infty} S(n)/n = 1$. Partial summation with $f(n) = 1/n$ gives

$$\sum_{n=1}^{N} a_n \frac{1}{n} = S(N)\frac{1}{N} - \sum_{n=1}^{N-1} S(n)\Big(\frac{1}{n+1} - \frac{1}{n}\Big).$$

Rewriting this, we have

$$\sum_{n=1}^{N-1} S(n)\Big(\frac{1}{n}-\frac{1}{n+1}\Big) = -S(N)\frac{1}{N}+\sum_{n=1}^{N} a_n\frac{1}{n} = \frac{O(1)}{N}+\sum_{n=1}^{N} a_n\frac{1}{n}.$$

Since $\frac{O(1)}{N}$ vanishes as N tends to infinity, we only need to worry about the left-hand side. We can rewrite it as

$$\sum_{n=1}^{N-1}\frac{S(n)}{n}\frac{1}{n+1}.$$

We now use the fact that $S(n)/n \to 1$. Fix $\varepsilon > 0$. Then we can pick M such that whenever $n \geq M$ we have $|S(n)/n - 1| < \varepsilon$. Hence for any $N > M$ we have

$$\begin{aligned} &\Big|\sum_{n=1}^{N-1}\frac{S(n)}{n}\frac{1}{n+1}-\sum_{n=1}^{N-1}\frac{1}{n+1}\Big| \\ &\leq \; C(M)+\Big|\sum_{n=M}^{N-1}\frac{S(n)}{n}\frac{1}{n+1}-\sum_{n=M}^{N-1}\frac{1}{n+1}\Big| \\ &\leq \; C(M)+\sum_{n=M}^{N-1}\frac{\varepsilon}{n+1} \\ &\leq \; C(M)+\varepsilon\log N. \end{aligned}$$

where $C(M) = |\sum_{n=1}^{M-1}\frac{S(n)}{n}\frac{1}{n+1}-\sum_{n=1}^{M-1}\frac{1}{n+1}|$. Divide both sides by $\log N$ to get

$$\Big|\frac{1}{\log N}\sum_{n=1}^{N-1}\frac{S(n)}{n}\frac{1}{n+1}-\frac{1}{\log N}\sum_{n=1}^{N-1}\frac{1}{n+1}\Big| < C(M)/\log N+\varepsilon.$$

Here, as $N \to \infty$, the right-hand side goes to ε, and $\frac{1}{\log N}\sum_{n=1}^{N-1}\frac{1}{n+1}$ approaches 1. Since ε is arbitrarily small, we conclude that $\frac{1}{\log N}\sum_{n=1}^{N-1}\frac{S(n)}{n}\frac{1}{n+1}$ approaches 1, too. Recalling the equality from the partial summation formula above, this immediately gives

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N} a_n\frac{1}{n} = 1$$

as desired.

The converse is false. For a counterexample, consider a sequence $\{a_n\}$ defined by $a_n = 2^k$ if $n = 2^k$, and $a_n = 1$ otherwise. Then

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{a_n}{n} = \frac{1}{\log N}\sum_{n\leq N, n\neq 2^k}\frac{1}{n}+\frac{1}{\log N}\sum_{n\leq N, n=2^k} 1.$$

When N is large, the first sum here is about $\frac{1}{\log N}\sum_{n\leq N}\frac{1}{n}$, because $\frac{1}{\log N}\sum_{n\leq N,n=2^k}\frac{1}{n} < \frac{1}{\log N}\cdot 2 \to 0$ as $N\to\infty$. And the second sum equals $\frac{1}{\log N}[\frac{\log N}{\log 2}]$, which converges to $\log 2$ as $N\to\infty$. Hence

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{a_n}{n} = 1+\log 2.$$

But then consider

$$\frac{1}{N}\sum_{n=1}^{N}a_n = \frac{1}{N}\sum_{n\leq N,n\neq 2^k}1 + \frac{1}{N}\sum_{n\leq N,n=2^k}n.$$

Again, when N is large, the first sum here is very close to $\frac{1}{N}\sum_{n\leq N}1 = 1$. But the second sum here oscillates. When $N = 2^k$, it equals $\frac{2^{k+1}-1}{2^k}$, which is about 2. But when $N = 2^k - 1$, it's $\frac{2^k-1}{2^k}$, about 1. So the sum oscillates between 1 and 2. That is, the limit doesn't even exist.

5. Let $N > 2$ be an integer. Set $a_n = 1$ and $f(n) = 1/\log n$ and do partial summation from 2 to N:

$$\sum_{n=2}^{N}\frac{1}{\log n} = \frac{N}{\log N} - \frac{1}{\log 2} - \sum_{n=2}^{N-1}n\Big(\frac{1}{\log(n+1)} - \frac{1}{\log n}\Big).$$

Two lemmas are in order. First is that

$$\begin{aligned}
\frac{N}{\log N} - \frac{2}{\log 2} &= \int_2^N \Big(\frac{t}{\log t}\Big)' dt \\
&= \int_2^N \frac{1}{\log t}dt - \int_2^N \frac{1}{(\log t)^2}dt
\end{aligned}$$

and the second is

$$\begin{aligned}
-\sum_{n=2}^{N-1}n\Big(\frac{1}{\log(n+1)} - \frac{1}{\log n}\Big) &= -\sum_{n=2}^{N-1}n\int_n^{n+1}\Big(\frac{1}{\log t}\Big)' dt \\
&= \sum_{n=2}^{N-1}n\int_n^{n+1}\frac{1}{t(\log t)^2}dt \\
&= \int_2^N \frac{[t]}{t(\log t)^2}dt \\
&= \int_2^N \frac{1}{(\log t)^2}dt - \int_2^N \frac{\{t\}}{t(\log t)^2}dt.
\end{aligned}$$

Consequently, we have

$$\sum_{n=2}^{N} \frac{1}{\log n} = \int_2^N \frac{1}{\log t}dt + \frac{1}{\log 2} - \int_2^N \frac{\{t\}}{t(\log t)^2}dt.$$

Let's show that

$$\int_2^N \frac{\{t\}}{t(\log t)^2}dt = O(\frac{1}{\log N}) + C_0$$

for some constant C_0. Note that

$$\int_2^\infty \frac{\{t\}}{t(\log t)^2}dt$$

is convergent because the integrand is bounded by $1/t(\log t)^2$ and $\int_2^\infty 1/t(\log t)^2 dt = 1/\log 2$ is convergent. Hence we can write

$$\int_2^N \frac{\{t\}}{t(\log t)^2}dt = \int_2^\infty \frac{\{t\}}{t(\log t)^2}dt - \int_N^\infty \frac{\{t\}}{t(\log t)^2}dt.$$

But the last term on the right is $O(1/\log N)$ because it is bounded by $\int_N^\infty 1/t(\log t)^2 dt = 1/\log N$. Therefore we have shown that

$$\sum_{n=2}^{N} \frac{1}{\log n} = \int_2^N \frac{1}{\log t}dt + C + O(\frac{1}{\log N})$$

for some constant C.

In general, for a real number $x > 2$, we have

$$\begin{aligned}
\sum_{2\le n\le x} \frac{1}{\log n} &= \int_2^{[x]} \frac{1}{\log t}dt + C + O(\frac{1}{\log[x]}) \\
&= \int_2^x \frac{1}{\log t}dt - \int_{[x]}^x \frac{1}{\log t}dt + C + O(\frac{1}{\log[x]}) \\
&= \int_2^x \frac{1}{\log t}dt + C + O(\frac{1}{\log x})
\end{aligned}$$

as desired.