MATH 152 Problem set 1 solutions

1. Factorize $n^4 + n^2 + 1$:

$$\begin{aligned} n^4 + n^2 + 1 &= n^4 + 2n^2 + 1 - n^2 \\ &= (n^2+1)^2 - n^2 \\ &= (n^2+n+1)(n^2-n+1). \end{aligned}$$

Since both factors are greater than 1 when $n > 1$, it follows that it is not prime.

2. (i) Proof by contradiction: suppose $\sqrt{p}$ is rational. Then we can write $\sqrt{p} = \frac{a}{b}$, where a, b are positive integers. Squaring both sides, we obtain

$$p = \frac{a^2}{b^2} = \left(\frac{a}{b}\right)^2.$$

This implies that $\frac{a}{b}$ is an integer (otherwise, then $b \nmid a$, so $b^2 \nmid a^2$, and consequently p is not an integer, a contradiction). Therefore p is a square; in particular, it has at least three different divisors. But this contradicts the assumption that p is prime.

(ii) We prove the contrapositive statement. Suppose $\sqrt{n}$ is rational, i.e. we can write $\sqrt{n} = \frac{a}{b}$ for some positive integers a, b. As before, square both sides to get

$$n = \frac{a^2}{b^2} = \left(\frac{a}{b}\right)^2.$$

Again $\frac{a}{b}$ has to be an integer. Therefore n is a square of an integer.

(iii) Suppose α is not irrational. Then we want to show that α is an integer. To be more precise, we show that $\alpha \notin \mathbb{Q} \setminus \mathbb{Z}$. Write $\alpha = \frac{r}{s}$ where $r, s \in \mathbb{Z}$ and $(r, s) = 1$. Then

$$\left(\frac{r}{s}\right)^n + a_1\left(\frac{r}{s}\right)^{n-1} + \ldots + a_n = 0.$$

Multiplying both sides by s^n:

$$r^n + a_1 r^{n-1} s + \ldots + a_n s^n = 0.$$

Therefore

$$r^n = -(a_1 r^{n-1} s + \ldots + a_n s^n).$$

The right-hand side here is divisible by s, and therefore so is the left-hand side r^n. But since $(r, s) = 1$, $(r^n, s) = 1$. Hence $s = \pm 1$ is forced.

3. Let p be a prime. The strategy is to compare the power of p in the factorization of the numerator with that of the denominator, and show that the former is no less than the latter for any prime p.

The power of p in the numerator is

$$\sum_{m=1}^{\infty}\Big(\Big\lfloor\frac{30n}{p^m}\Big\rfloor+\Big\lfloor\frac{n}{p^m}\Big\rfloor\Big).$$

(Here $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.)

And the power of p in the denominator is

$$\sum_{m=1}^{\infty}\Big(\Big\lfloor\frac{15n}{p^m}\Big\rfloor+\Big\lfloor\frac{10n}{p^m}\Big\rfloor+\Big\lfloor\frac{6n}{p^m}\Big\rfloor\Big).$$

It suffices to show that $\lfloor\frac{30n}{p^m}\rfloor+\lfloor\frac{n}{p^m}\rfloor\geq\lfloor\frac{15n}{p^m}\rfloor+\lfloor\frac{10n}{p^m}\rfloor+\lfloor\frac{6n}{p^m}\rfloor$ for each m. To simplify notation a little bit, let $\frac{n}{p^m}=N+\alpha$, where $N=\lfloor\frac{n}{p^m}\rfloor$ and $\alpha=\frac{n}{p^m}-N$. Note that here $0\leq\alpha<1$. Then $\frac{30n}{p^m}=30N+30\alpha$, which gives $\lfloor\frac{30n}{p^m}\rfloor=\lfloor 30\alpha\rfloor$. By the same logic, $\lfloor\frac{15n}{p^m}\rfloor=\lfloor 15\alpha\rfloor$, $\lfloor\frac{10n}{p^m}\rfloor=\lfloor 10\alpha\rfloor$, $\lfloor\frac{6n}{p^m}\rfloor=\lfloor 6\alpha\rfloor$. So all we really need to show is

$$\lfloor 30\alpha\rfloor+\lfloor\alpha\rfloor\geq\lfloor 15\alpha\rfloor+\lfloor 10\alpha\rfloor+\lfloor 6\alpha\rfloor$$

for any $0\leq\alpha<1$.

We divide the proof into 30 cases: for each $a\in\{0,1,2,\ldots,29\}$, we show that for $a/30\leq\alpha<(a+1)/30$ the above inequality holds. The verification of this claim is easy and not as tedious as it seems, it is left to the reader. To give an idea, the first few cases go like:

Case $a=0$: $\lfloor 30\alpha\rfloor=\lfloor 15\alpha\rfloor=\lfloor 10\alpha\rfloor=\lfloor 6\alpha\rfloor=0$.

Case $a=1$: $\lfloor 30\alpha\rfloor=1$, and $\lfloor 15\alpha\rfloor=\lfloor 10\alpha\rfloor=\lfloor 6\alpha\rfloor=0$.

Case $a=2$: $\lfloor 30\alpha\rfloor=2$, $\lfloor 15\alpha\rfloor=1$, $\lfloor 10\alpha\rfloor=\lfloor 6\alpha\rfloor=0$.

Case $a=3$: $\lfloor 30\alpha\rfloor=3$, $\lfloor 15\alpha\rfloor=1$, $\lfloor 10\alpha\rfloor=1$, $\lfloor 6\alpha\rfloor=0$.

Case $a=4$: $\lfloor 30\alpha\rfloor=4$, $\lfloor 15\alpha\rfloor=2$, $\lfloor 10\alpha\rfloor=1$, $\lfloor 6\alpha\rfloor=0$.

...and so on. In general, $\lfloor 15\alpha\rfloor$ increases by 1 as a increases by 2, $\lfloor 10\alpha\rfloor$ increases as a increases by 3, and $\lfloor 6\alpha\rfloor$ increases by 1 as a increases by 5.

4. Suppose first that $a \mid bc$. Write $a = a'(a,b)$ and $b = b'(a,b)$. Note that $(a',b') = 1$. Then $a'(a,b) \mid b'(a,b)c \Rightarrow a' \mid b'c$, but since $(a',b') = 1$ we have $a' \mid c$.

Conversely, if $a' \mid c$ then $a' \mid b'c$ so $a \mid bc$.

5. Pick $1 \leq a < b \leq n$. Then $(an!+1, bn!+1) = (an!+1, (b-a)n!)$. But then any prime divisor of $an!+1$ is greater than n, whereas any prime divisor of $(b-a)n!$ is no greater than n. Therefore $(an!+1, bn!+1) = 1$.

As a consequence, for any n we can find n distinct integers greater than 1 that are pairwise coprime. But if there were no more than, say, $N < \infty$ primes, then any $N+1$ numbers greater than 1 is not pairwise coprime, by the pigeonhole principle. (To elaborate, make N boxes corresponding to the N primes and "put" a number into a box corresponding to any of its prime divisors. If we put $N+1$ numbers in, then at least one box must have at least two numbers in it.)

6. We are asked to compute the maximum power of 10 dividing 2010!. To do this we compute the maximum power of 2 and 5 dividing 2010! and take the smaller of the two. The former equals

$$\sum_{i=1}^{10} \left\lfloor \frac{2010}{2^i} \right\rfloor = 2002$$

and the latter is

$$\sum_{i=1}^{4} \left\lfloor \frac{2010}{5^i} \right\rfloor = 501.$$

Therefore the answer is 501.