# MATH 152: PROBLEM SET 9

Due December 1

1. (a) Let $q$ be a natural number and let $c_1, \ldots, c_q$ be arbitrary complex numbers. Prove that

$$\sum_{\chi \pmod q} \Big| \sum_{n=1}^{q} c_n \chi(n) \Big|^2 = \phi(q) \sum_{\substack{n=1 \\ (n,q)=1}}^{q} |c_n|^2.$$

(b). Let $q$ be a natural number and for each character $\chi \pmod q$ let $c_\chi$ be some complex number. Prove that

$$\sum_{n=1}^{q} \Big| \sum_{\chi \pmod q} c_\chi \chi(n) \Big|^2 = \phi(q) \sum_{\chi \pmod q} |c_\chi|^2.$$

2. Let $\chi \pmod q$ be a non-principal character. Prove that as $x \to \infty$

$$\sum_{n \le x} d(n)\chi(n) = O(\sqrt{x}),$$

where $d(n) = \sum_{d|n} 1$ counts the number of divisors of $n$.

3. Let $p$ be an odd prime and in this problem $\chi$ will denote the Legendre symbol $\pmod p$.

(i). Prove that

$$\frac{\zeta(4)}{\zeta(2)} \le L(2,\chi) \le \zeta(2).$$

(ii). Given any $\epsilon > 0$ prove that there are infinitely many primes $p$ such that

$$L(2,\chi) \ge \zeta(2) - \epsilon.$$

Prove also that there are infinitely many primes $p$ such that

$$L(2,\chi) \le \frac{\zeta(4)}{\zeta(2)} + \epsilon.$$

4. (a) Given a natural number $q$ prove that

$$\sum_{\substack{n \le x \\ (n,q)=1}} 1 = \frac{\phi(q)}{q} x + O(\phi(q)).$$

(b). We say that a set $\mathcal{A}$ of natural numbers has *density* $\alpha$ if

$$\lim_{x\to\infty} \frac{1}{x} \#\{a \le x : \quad a \in \mathcal{A}\}$$

exists and equals $\alpha$. Let $\mathcal{B}$ denote the set of natural numbers $n$ such that 2010 divides $\phi(n)$. Prove that $\mathcal{B}$ has density 1 (or, equivalently that the set of natural numbers $n$ such that $2010 \nmid \phi(n)$ has density zero).