MATH 152: PROBLEM SET 8

Due November 17

1. Use partial summation to show that

$$\sum_{n\le N}\log n = N\log N - N + 1 + \int_1^N \frac{\{t\}}{t}dt.$$

Consider $F(x) = \int_1^x \{t\}dt$. What can you say about $F(x)$ for large x? Use that information, and integration by parts to establish that

$$\int_1^N \frac{\{t\}}{t}dt = \frac{1}{2}\log N + C_0 + O\Big(\frac{1}{N}\Big),$$

for some constant C_0. Conclude that $N! \sim C\sqrt{N}(N/e)^N$ for some constant $C > 0$. (In fact, $C = \sqrt{2\pi}$, and this is Stirling's formula for $N!$.)

2. Euler made his name by showing that $1/1 + 1/4 + 1/9 + \ldots = \pi^2/6$. Once he found the answer he was able to check it was right by computing the series accurately. If you just sum the series up to N terms, what would be the error from the true answer? Can you think of a better way to estimate/evaluate the tail

$$\sum_{n=N+1}^{\infty}\frac{1}{n^2}?$$

This is a little vague: what I have in mind is that if you do the obvious estimation, then after summing the first 1000 terms you'd have an error of about 10^{-3}. After the refined estimation, you should have an error of about 10^{-6}.

3. Prove that for $s > 1$

$$\zeta(s) = \frac{s}{(s-1)} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}dt.$$

For what values of s does the integral above make sense? This can be used to make sense of $\zeta(s)$ for values of s other than $s > 1$.

4. Let a_n be a sequence of complex numbers such that

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}a_n = 1.$$

Prove using partial summation, or otherwise, that

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\leq x} \frac{a_n}{n} = 1.$$

Is the converse true? Prove or give a counterexample.

5. Prove that as $x \to \infty$

$$\sum_{2\leq n\leq x} \frac{1}{\log n} = \int_2^x \frac{dt}{\log t} + C + O\Big(\frac{1}{\log x}\Big),$$

for some constant C.