MATH 152: PROBLEM SET 6

Due November 3

1. Develop the arithmetic of the ring $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} : \quad a, b \in \mathbb{Z}\}$. Show that there are division and Euclidean algorithms for this ring. What are the irreducibles and primes in this ring; are they the same? Explain why unique factorization holds.

2. Use your work in question 2 to describe what primes are of the form $x^2 + 2y^2$. More generally, what numbers may be written as $x^2 + 2y^2$?

3. In the spirit of Euclid, give elementary proofs that there are infinitely many primes $p \equiv 1 \pmod 3$, and $p \equiv 2 \pmod 3$.

4. Define the multiplicative function $\mu(n)$ (the Möbius function by setting $\mu(1) = 1$, $\mu(p) = -1$ (for primes p) and $\mu(p^k) = 0$ for $k > 1$. When does the series $\sum_{n=1}^{\infty} \mu(n) n^{-s}$ converge? Prove that in the range $s > 1$ we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

and that

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^s} = \frac{\zeta(s)}{\zeta(2s)}.$$

5. Let $d(n)$ denote the number of divisors d of n.

(i) Prove that $d(n)$ is a multiplicative function: that is $d(mn) = d(m)d(n)$ if m and n are coprime. Prove that $d(p^a) = a + 1$ for any prime power p^a.

(ii) Prove that for any $\epsilon > 0$ there exists a constant $C(\epsilon)$ such that

$$d(n) \le C(\epsilon) n^{\epsilon}$$

for all n. Suggestion: if $n = \prod_i p_i^{a_i}$ then

$$\frac{d(n)}{n^{\epsilon}} = \prod_i \frac{(a_i + 1)}{p_i^{a_i \epsilon}}.$$

What can you say about the terms $(a_i + 1) p_i^{-a_i \epsilon}$?

(iii) For what real numbers s does the series $\sum_{n=1}^{\infty} d(n)/n^s$ converge? In the range of convergence, can you identify this function with something in terms of the Riemann zeta function?