MATH 152: PROBLEM SET 2

Due October 6

1. Let $p \neq 2, 5$ be a prime. The decimal expansion of $1/p$ will involve a certain number of digits which repeat. (E.g. $1/3 = 0.3333..$ has one digit repeating, and $1/7$ has six digits repeating.) Prove that the number of digits that repeat equals the order of $10 \pmod p$.

2. Let d_N denote the least common multiple of the first N natural numbers 1, 2, $\ldots$, N. Let $\pi(x)$ denote the number of primes p with $p \leq x$.

(a) What is the power of p dividing d_N? Prove that

$$\log d_N = \sum_{p \leq N} \log p \left[\frac{\log N}{\log p} \right] \leq (\log N) \pi(N).$$

(b) Let $f(x) = \sum_i a_i x^i$ be a polynomial with integer coefficients and with degree $\leq N - 1$. Prove that

$$d_N \int_0^1 f(x) dx \in \mathbb{Z}.$$

(c) Take $f_N(x) = x^N (1-x)^N$ and use (b) to show that $d_{2N+1} \int_0^1 f_N(x) dx \geq 1$.

(d) Show that $\int_0^1 f_N(x) dx \leq 4^{-N}$ and using (c) and (a) deduce that

$$\pi(2N+1) \geq \frac{(2 \log 2) N}{\log(2N+1)}.$$

3. Given any number n prove that there are only finitely many natural numbers x such that $\phi(x) = n$. Find all integers with x with $\phi(x) = 100$.

4. If p is a prime prove that $(p-1)! \equiv (p-1) \pmod{1 + 2 + 3 + \ldots + (p-1)}$.

5. Let p be an odd prime, and let a be coprime to p. If $a \not\equiv 1 \pmod p$, prove that p divides $1 + a + a^2 + \ldots + a^{p-2}$.

6. Let $f(x)$ be a polynomial with integer coefficients. If $f(a) \equiv k \pmod m$ show that $f(a + tm) \equiv k \pmod m$ for all integers t. Using this show that $f(x)$ cannot be prime for all integer values of x.