BERTRAND'S POSTULATE

For every natural number $n \geq 2$, Bertrand's postulate says that there is a prime between n and $2n$. Bertrand checked this numerically for many values of n, but the result was first established by the Russian mathematician Chebyshev in 1850. We give a proof due to Paul Erdős which builds upon an idea of Ramanujan.

The main idea is to look at the prime factorization of the binomial coefficient $\binom{2n}{n}$. We first record what this factorization looks like.

Proposition 1. *In the prime factorization of $\binom{2n}{n}$, the prime p appears to the power*

$$\sum_{k=1}^{\infty} \Big([\frac{2n}{p^k}] - \Big[\frac{n}{p^k}\Big] \Big).$$

Note only primes below $2n$ appear in the factorization. Every prime in $[n+1, 2n)$ appears to the exponent 1. If $n \geq 5$, no prime in $(2n/3, n]$ can divide $\binom{2n}{n}$. Any prime $p > \sqrt{2n}$ appears to exponent 0 or 1, and a prime $p \leq \sqrt{2n}$ appears to exponent at most $\log(2n)/\log p$.

Proof. Recall that the power of p that divides $n!$ is $\sum_{k=1}^{\infty}[n/p^k]$. Therefore, the power of p that divides $\binom{2n}{n}$ is

$$\sum_{k=1}^{\infty} \Big(\Big[\frac{2n}{p^k}\Big] - 2\Big[\frac{n}{p^k}\Big] \Big). \tag{1}$$

Note that although we wrote an infinite sum above, only finitely many terms are non-zero. Also note that $[2x] - 2[x]$ takes only the values 0 (if the fractional part of x is $< 1/2$) and 1 (if the fractional part is $\geq 1/2$). If $p > \sqrt{2n}$ then only the term $k = 1$ in (1) can be non-zero, and so such a prime appears to exponent 0 or 1. If $p \leq \sqrt{2n}$ then only the terms with $1 \leq k \leq \log(2n)/\log p$ can be non-zero in (1), and so such a prime appears at most to the exponent $\log(2n)/\log p$. We have justified the last assertion in our Proposition.

To justify the first two, note that if $2n \geq p \geq n+1 (> \sqrt{2n})$ then only the term $k = 1$ in (1) matters, and $[2n/p] - 2[n/p] = 1 - 0 = 1$. If $n \geq 5$ and $n \geq p > 2n/3 > \sqrt{2n}$, again only $k = 1$ matters and here $[2n/p] - 2[n/p] = 2 - 2 = 0$. This completes our proof.

Next we give a lower bound for the size of the middle binomial coefficient $\binom{2n}{n}$.

Proposition 2. *For $n \geq 1$ we have*

$$\binom{2n}{n} \geq \frac{2^{2n}}{2n}.$$

Proof. If $n \geq 1$ then the middle binomial coefficient is the largest of the binomial coefficients $\binom{2n}{j}$, and moreover it is at least $2 = \binom{2n}{0} + \binom{2n}{2n}$. Thus

$$\binom{2n}{n} \geq \frac{1}{2n}\left(\left\{\binom{2n}{0} + \binom{2n}{2n}\right\} + \binom{2n}{1} + \ldots + \binom{2n}{2n-1}\right) = \frac{2^{2n}}{2n}.$$

Proposition 3. *For all real numbers $x \geq 1$ we have*

$$\prod_{p \leq x} p \leq 4^x.$$

Granting for the moment Proposition 3, let us now prove Bertrand's postulate.

Theorem. *For every $n \geq 1$ there is a prime in $[n+1, 2n]$.*

Proof. Let us suppose that $n \geq 500$, and that there is no prime in $[n+1, 2n]$. By Propositions 1 and 2 we have that

$$\frac{2^{2n}}{2n} \leq \binom{2n}{n} \leq \prod_{p \leq \sqrt{2n}} p^{\log(2n)/\log p} \prod_{\sqrt{2n} < p \leq 2n/3} p,$$

where in the upper bound above we used that there are no primes in $[n+1, 2n]$ and that no prime in $(2n/3, n]$ can divide $\binom{2n}{n}$. Using Proposition 3, we obtain that

$$\frac{2^{2n}}{2n} \leq \prod_{p \leq \sqrt{2n}} (2n) \times 4^{2n/3} = (2n)^{\pi(\sqrt{2n})} 4^{2n/3},$$

or simplifying that

$$2^{2n/3} \leq (2n)^{\pi(\sqrt{2n})+1} < (2n)^{\sqrt{2n}}.$$

Using calculus you can check that this inequality cannot hold if $n \geq 500$. Thus Bertrand's postulate must be true for $n \geq 500$.

Note that

$$2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631$$

is a sequence of prime numbers each successive term of which is less than twice the previous one. This verifies Bertrand's postulate for n up to 500.

Proof of Proposition 3. It suffices to establish the Proposition when x is an integer. Clearly the result is true for $x = 1$ and $x = 2$. Now suppose the result holds for all integers 1, 2, ..., $x - 1$ and we want to establish it for x. If $x \geq 4$ is even then x is not prime, and $\prod_{p \leq x} p = \prod_{p \leq x-1} p \leq 4^{x-1} < 4^x$.

Now suppose that $x = 2n + 1$ is odd. Arguing as in Proposition 1 we may easily see that every prime p in $[n+2, 2n+1]$ divides the binomial coefficient $\binom{2n+1}{n}$. Therefore, using our induction hypothesis,

$$\prod_{p \leq 2n+1} p = \prod_{p \leq n+1} p \times \prod_{n+2 \leq p \leq 2n+1} p \leq 4^{n+1} \times \binom{2n+1}{n}. \tag{2}$$

Now $\binom{2n+1}{n} = \binom{2n+1}{n+1}$ and so

$$2\binom{2n+1}{n} = \binom{2n+1}{n} + \binom{2n+1}{n+1} < \binom{2n+1}{0} + \ldots + \binom{2n+1}{2n+1} = 2^{2n+1},$$

or in other words, $\binom{2n+1}{n} \leq 2^{2n}$. Inserting this in (2) we conclude that

$$\prod_{p \leq 2n+1} p \leq 4^{n+1} \times 4^{n} = 4^{2n+1},$$

which establishes our induction step, and hence Proposition 3.