

MASTERCLASS WEEK 2: PROBLEMS

1. (Daniel Le) Let a_n ($n \geq 1$) be an increasing sequence of positive real numbers such that $\lim_{n \rightarrow \infty} a_n/n = 0$. Must there exist infinitely many positive integers n such that $a_{n-i} + a_{n+i} < 2a_n$ for $i = 1, 2, \dots, n-1$? (2001: B6)
2. (Jeffrey Wang) Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=1}^{\infty} a_n x^n.$$

Prove that for each integer $n \geq 0$ there exists an integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

(1999: A3)

3. (Nathan Pflueger) Let α be a real number such that $1^\alpha, 2^\alpha, 3^\alpha, \dots$ are all natural numbers. Show that α is a non-negative integer.
4. (Boris Hanin; USAMTS, 2002-2003) A fudgeflake is a planar fractal figure with 120 degree rotational symmetry such that three identical fudgeflakes in the same orientation fit together without gaps to form a larger fudgeflake with its orientation 30 degrees clockwise of the smaller fudgeflakes' orientation (Boris will draw a figure!). If the distance between the centers of the original three fudgeflakes is 1 what is the area of one of those three fudgeflakes?
5. (Sound) Among the numbers 2^n ($1 \leq n \leq 10^6$) how many begin with the leading decimal digit 1? Among the numbers 2^n , which leading digit appears more frequently 7 or 8?