MATH 152: MIDTERM EXAMINATION

Due Thursday, December 11 at Noon in my office (383 W)

The Rules

You are free to consult your class notes, and the books by Niven-Zuckerman-Montgomery, Hardy-Wright, Stark, and LeVeque placed on reserve at the library. You may use any result that we have covered so far in class, but you must state clearly what you are using. You may not discuss the exam with others, nor use the internet or any other source, with the exception of the books listed above. You should write out and sign the honor code, indicating also your acceptance of these rules. All the best!

The Problems

1. (a). (10 points) Let q be a natural number and let $c_1, \ldots, c_q$ be arbitrary complex numbers. Prove that

$$\sum_{\chi \pmod q} \Big| \sum_{n=1}^{q} c_n \chi(n) \Big|^2 = \phi(q) \sum_{\substack{n=1 \\ (n,q)=1}}^{q} |c_n|^2.$$

(b). (10 points) Let q be a natural number and for each character $\chi \pmod q$ let c_χ be some complex number. Prove that

$$\sum_{n=1}^{q} \Big| \sum_{\chi \pmod q} c_\chi \chi(n) \Big|^2 = \phi(q) \sum_{\chi \pmod q} |c_\chi|^2.$$

2. (15 points) Let $\chi \pmod q$ be a non-principal character. Prove that as $x \to \infty$

$$\sum_{n \le x} d(n)\chi(n) = O(\sqrt{x}),$$

where $d(n) = \sum_{d|n} 1$ counts the number of divisors of n.

3. Let p be an odd prime and in this problem χ will denote the Legendre symbol $\pmod p$.

(i). (5 points) Prove that

$$\frac{\zeta(4)}{\zeta(2)} \le L(2,\chi) \le \zeta(2).$$

(ii). (15 points) Given any $\epsilon > 0$ prove that there are infinitely many primes p such that

$$L(2, \chi) \geq \zeta(2) - \epsilon.$$

Prove also that there are infinitely many primes p such that

$$L(2, \chi) \leq \frac{\zeta(4)}{\zeta(2)} + \epsilon.$$

4. (a) (5 points) Given a natural number q prove that

$$\sum_{\substack{n \leq x \\ (n,q)=1}} 1 = \frac{\phi(q)}{q} x + O(\phi(q)).$$

(b) (15 points) We say that a set $\mathcal{A}$ of natural numbers has *density* α if

$$\lim_{x \to \infty} \frac{1}{x} \#\{a \leq x : \quad a \in \mathcal{A}\}$$

exists and equals α. Let N be a positive square-free number and consider the quadratic form $x^2 + Ny^2$. Let $\mathcal{B}$ denote the set of integers properly represented by this quadratic form. Show that $\mathcal{B}$ has density zero. (Hint: consider first the simple case $N = 1$ and try to generalize.)

5. (a) (8 points) Determine the class number of the discriminant $D = -132$. List the reduced forms for this discriminant.

(b) (8 points) Given a prime $p \neq 2$, 3 or 11, give explicit congruence conditions on p (mod 132) which describe whether p must be properly represented by some form of discriminant -132.

(c) (9 points) Determine the prime numbers that are represented by $x^2 + 33y^2$.