

## MATH 152: FINAL EXAM

DUE JUNE 11 BY NOON, IN MY OFFICE (383W)

### THE RULES

You are free to consult your class notes, Landau's book and the books by Niven-Zuckerman-Montgomery, and Hardy-Wright placed on reserve at the library. You may use any result that we have covered so far in class, but you must state clearly what you are using. You may not discuss the exam with others, nor use the internet or any other source, with the exception of the books listed above. You should write out and sign the honor code, indicating also your acceptance of these rules. All the best!

### GRADING POLICY

Recall that the final is worth 40% of your grade. You could score a maximum of 120 points on this final: 100 regular, and 20 extra credit. I will determine everyone's grades first based only on the regular points. Then I will use the extra credit points to move up grades. In other words, your grade will not be adversely affected by not attempting the extra credit problems (and even if everyone else does them). Of course, I expect that it'll be too much fun for you to resist!

### THE PROBLEMS

- (10 points) For every prime  $2 \leq \ell \leq 97$  you are given a choice of sign  $\epsilon_\ell = +1$  or  $-1$ . Show that you can find a prime  $p$  such that  $\left(\frac{\ell}{p}\right) = \epsilon_\ell$  for each prime  $\ell$  with  $2 \leq \ell \leq 97$ .
- Let  $p$  be an odd prime throughout this problem.
  - (5 points) Let  $\chi \pmod{q}$  be a character. Show that

$$\sum_{n \pmod{p}} \overline{\chi(n)} \chi(n+1) = \sum_{n=1}^{p-1} \chi(1 + \bar{n}),$$

where  $\bar{n}$  denotes the multiplicative inverse of  $n$ : that is,  $n\bar{n} \equiv 1 \pmod{p}$ . Complete the evaluation of the sum above.

- (15 points) Evaluate

$$\sum_{n \pmod{p}} \left( \frac{n(n+1)}{p} \right).$$

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How many numbers  $n$  are there with  $1 \leq n \leq p$  such that both  $n$  and  $n + 1$  are quadratic residues  $(\text{mod } p)$ ? How many  $n$  are there with  $1 \leq n \leq p$  such that both  $n$  and  $n + 1$  are quadratic non-residues?

(c). (Extra credit: 10 points) Explain how you would go about evaluating (given  $a, b, c$ )

$$\sum_{n \pmod{p}} \left( \frac{an^2 + nb + c}{p} \right).$$

3. (a). (5 points) Given a natural number  $q$ , prove that

$$\sum_{\substack{n \leq x \\ (n, q) = 1}} 1 = \frac{\phi(q)}{q}x + O(\phi(q)).$$

(b). (15 points) We say that a set  $\mathcal{A}$  of natural numbers has *density*  $\alpha$  if

$$\lim_{x \rightarrow \infty} \frac{1}{x} \#\{a \leq x : a \in \mathcal{A}\}$$

exists and equals  $\alpha$ . Let  $\mathcal{B}$  denote the set of natural numbers  $n$  such that 2007 divides  $\phi(n)$ . Prove that  $\mathcal{B}$  has density 1 (or, equivalently that the set of natural numbers  $n$  such that  $2007 \nmid \phi(n)$  has density zero).

4. (10 points) Let  $f(x, y) = ax^2 + bxy + cy^2$  be a binary quadratic form of discriminant  $D$ . Let  $p$  be an odd prime dividing  $D$ . Show that the values taken by  $f(x, y) \pmod{p}$  are either

- (i) always 0  $(\text{mod } p)$ ; or
- (ii) 0  $(\text{mod } p)$ , and all quadratic residues  $(\text{mod } p)$ ; or
- (iii) 0  $(\text{mod } p)$ , and all quadratic non-residues  $(\text{mod } p)$ .

5. (a). (5 points) Determine the class number for the discriminant  $D = -84$  and list the reduced forms.

(b). (8 points) You are given primes  $p_1, p_2, p_3$ , and  $p_4$  such that  $p_1 \equiv 31 \pmod{84}$ ,  $p_2 \equiv 37 \pmod{84}$ ,  $p_3 \equiv 41 \pmod{84}$ , and  $p_4 \equiv 43 \pmod{84}$ . Explain why each of  $p_1, p_2$  and  $p_3$  must be represented by one of your reduced forms from part (a), but why  $p_4$  will not be represented by any of those forms.

(c). (7 points) For each of the seven numbers  $p_1, p_2, p_3, p_1p_2, p_2p_3, p_1p_3, p_1p_2p_3$  determine which of the forms in part (a) represents that number.

6. In 1772 Euler found the remarkable polynomial  $n^2 + n + 41$  which is prime for  $n = 0, 1, \dots, 39$ . This problem will help you understand why this polynomial takes prime values.

(a). (5 points) What is the class number for the discriminant  $D = -163$ ? List the reduced forms. Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{n}{163} \right).$$

(b). (15 points) Let  $A$  be a positive odd number, and put  $D = 1 - 4A$ . Suppose that the class number for the discriminant  $D$  is 1. Prove that no odd prime  $p < A$  can be represented by  $x^2 + xy + Ay^2$ , and that  $\left(\frac{D}{p}\right) = -1$  for every prime  $2 < p < A$ . Deduce that  $n^2 + n + A$  is prime for  $n = 0, 1, \dots, A - 2$ .

(c). (Extra Credit 10 points) Show the converse to part (b). Namely, if  $n^2+n+A$  takes prime values for  $n = 0, 1, \dots, A-2$ , prove that the class number for the discriminant  $D = 1 - 4A$  is 1.