HW 7 Solutions

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Chapter 4.5 Exercise 29: If G is a non-abelian simple group of order < 100, prove that $G \approxeq A_5$.

Solution: OK, I'll be honest here. I don't particularly enjoy writing up these solutions, and this problem especially pisses me off. Forgive me if I take as many shortcuts as possible. Hopefully you've covered these cases either in class or you've read about them in your book. Groups of order p^α are abelian or not simple. Groups of order 12 and order 30 are not simple. If $p < q < r$ are primes, then groups of order pq, p^2q, pq^2 and pqr are not simple. Oh, and the book gives proof that any group of order 60 that is simple is isomorphic to A_5, so we just have to show that groups of order $24, 36, 40, 48, 56, 72, 80, 84, 88, 90, 96$, and 100 are abelian or not simple.

If I'm not mistaken, it turns out that all groups of order $p^\alpha q^\beta$ are abelian or not simple (Burnside's Theorem). If we assume this, we only have groups of order 84 and 90 to rule out...but this would be akin to cheating. It is good trivia to know anyway.

Some of these orders are quick and easy: 40, 84, 88, and 100. For instance, 40 equals $2^3 \cdot 5$. The number of Sylow 5-subgroups is congruent to 1 modulo 5 and divides 8. The only number that fits the bill is 1. So any group of order 40 has a normal subgroup isomorphic to $\mathbb{Z}/5$. An analogous argument shows that any group of order 88 must have a normal subgroup isomorphic to $\mathbb{Z}/11$ and any group of order 100 must have a normal subgroup or order 25. We get lucky on 84 as well, since its number of Sylow 7-subgroups is congruent to 1 modulo 7, but divides 12. Hence it has a normal subgroup isomorphic to $\mathbb{Z}/7$.

Proposition 0.1. *Let p and q be primes and suppose G is a group of order*

$p^\alpha q$ and the number of Sylow q-subgroups is p^α, then G must have exactly one Sylow p-subgroup (which must be normal by the Sylow Theorem).

Proof. The intersection of any two Sylow q-sugroups is the identity subgroup, so there must be $p^\alpha(q-1)$ elements in G of order q. This leaves p^α elements whose order is not equal to q. A Sylow p-subgroup has exactly that many elements, so evidently there can be at most one Sylow p-subgroup, in which case we're done. □

This rules out groups of order $80 = 2^4{\cdot}5$ and $56 = 2^3{\cdot}7$. In the case of 80 for instance, the number of Sylow 5-subgroups is congruent to 1 mod5 and divides 16. Either there is one Sylow 5-subgroup, which must be normal, or there are 16 of them, in which case we use the proposition to conclude there is one normal Sylow 2-subgroup. Groups of order 56 are dealt with analogously.

Now we're left to rule out groups of orders $24, 36, 48, 72, 90$, and 96. We'll need another little proposition.

Proposition 0.2. *Let G be a finite group whose order is divisible by p and let n_p be the number of Sylow p-subgroups of G. If $n_p! < |G|$ then G is not simple.*

Proof. G acts on the set of its Sylow p-subgroups by conjugation. After enumerating these subgroups, the action yields a group homomorphism

$$\phi : G \longrightarrow S_{n_p}$$

If $\ker\phi = G$, then $gPg^{-1} = P$ for every $g \in G$ and Sylow p-subgroup P, in particular they are all normal subgroups (well, there must only be one of them in this case). Since P is normal, G is not simple. Now suppose that $\ker\phi$ is a proper subgroup of G. As it is the kernel of a group homomorphism, it is a normal subgroup. We claim that it can't be the trivial group. Indeed we have

$$G/\ker\phi \approxeq \mathrm{Im}\phi \leq S_{n_p}.$$

Since the groups here are all finite, this implies that

$$\frac{|G|}{|\ker\phi|} \quad = \quad |\mathrm{Im}\phi| \leq n_p!|\ker\phi| = \frac{|G|}{|\mathrm{Im}\phi|} \geq \frac{|G|}{n_p!} > 1.$$

But since $|\ker\phi| > 1$, we have evidently found a normal, nontrivial, proper subgroup of G, so G can't be simple. □

This proposition takes care of groups of order 36 for example. The number of its Sylow 3-subgroups is congruent to 1mod3 and divides 4. Thus there are either 1 or 4 of them. In the former case, we're done immediately. In the latter case, we may apply the proposition because the hypothesis $n_p! < |G|$ is satisfied ($4! < 36$). An analogous argument works for groups of order 72 (again when $p = 3$) and groups of order $24, 48$, and 96 (using $p = 2$).

This leaves groups of order 90.

The number of Sylow 5-subgroups is congruent to 1mod5 and divides 18. This means there are either 1 or 6 such subgroups. If there is one of them, then we're done. If there are 6 of them, then since each Sylow 5-subgroup is isomorphic to $\mathbb{Z}/5$ their intersections must be the identity group. Thus there are $6 \cdot 4 = 24$ elements of order 5.

Now consider the number of Sylow 3-subgroups. This number is congruent to 1mod3 and divides 10. Evidently this number is either 1 or 10. If it is 1, we are done. If it is 10, then we have to think harder. If all of them have pairwise intersection equal to the identity group, then there would be $10 \cdot 8$ elements of G whose order is divisible by 3. But G only has 90 elements and we've already asked it to have $24 + 80 > 90$ elements. There must be two Sylow 3-subgroups P and Q whose intersection is nontrivial. Consider the normalizer of $P \cap Q$, $N(P \cap Q)$. Since P and Q are groups of order 9, you may check that $P \cap Q$ is normal in either of them (for instance all p-groups have nontrivial centers. Either $P \cap Q$ is the center, or P and Q are abelian). This implies that $P \leq N(P \cap Q)$ (and $Q \leq N(P \cap Q)$). By Lagrange's theorem, 9 divides the $|N(P \cap Q)|$ and $|N(P \cap Q)|$ divides 90. $|N(P \cap Q)|$ can't be 9 because $P \cup Q \subset N(P \cap Q)$ and $|P \cup Q| = 15$. If $|N(P \cap Q)| = 90$, then $P \cap Q$ is a normal subgroup of G in which case we're done. If $|N(P \cap Q)| = 45$ then $N(P \cap Q)$ is a subgroup of index 2 in G, and as such must be normal by an exercise from many moons ago. In this case we're done as well. This leaves the possibility that $|N(P \cap Q)| = 18$. But now we're almost done. After enumerating the five left cosets of $N(P \cap Q)$ in G, the action of G on the left cosets by left multiplication gives us a group homomorphism

$$\phi : G \longrightarrow S_5$$

We know that $\ker\phi \leq N(P \cap Q)$, so it is a proper, normal subgroup of G. If $\ker\phi$ were trivial, then we'd have $|\text{Im}\phi| = \frac{|G|}{\ker\phi} = \frac{90}{1} = 90$. On the otherhand, since $\text{Im}\phi$ is a subgroup of S_5, by Lagrange's theorem, we have 90 divides $5! = 120$ which is a contradiction. So if $N(P \cap Q)$ has order 18, then set of elements of G which act as the identity on the left cosets of $N(P \cap Q)$ form a normal, nontrivial, proper subgroup of G. Hence G is not simple.

Note: Why did we have to wait so long to use the last argument. We knew a long time ago there were Sylow 3-subgroups of order 9. Why didn't we act on their left cosets by left multiplication? Well, then we'd get a group homomorphism of G into S_6 and S_6 IS divisible by 90, so it could well be that the kernel of the map is trivial.

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Chapter 4.5 Exercise 30: How many elements of order 7 must there be in a simple group of order 168?

Solution: 168 factors as $2^3 \cdot 3 \cdot 7$, so the Sylow 7-subgroups have order 7. The union of the Sylow 7-subgroups consist of the elements of order 7 and the identity. For an element of order 7 generates a subgroup of order seven which is a Sylow 7-subgroup. Conversely any Sylow 7-subgroup has order 7, so it must be isomorphic to $\mathbb{Z}/7$ which consists of elements of order 7 and the identity element. Now the intersection of any two Sylow 7-subgroups must be just the identity subgroup by an application of Lagrange's theorem. This shows that the set of elements of order 7 is the disjoint union of the set of nonidentity elements of all the Sylow 7-subgroups. Each Sylow 7-subgroup contains 6 elements of order 7. So,

$$\# \text{ elements of order } 7 = 6 \cdot n_7$$

It remains to figure out the value of n_7. The Sylow Theorems tells us that $n_7 \equiv 1 \text{mod} 7$ and n_7 divides 24. There are only two options; n_7 equals 1 or n_7 equals 8. If $n_7 = 1$, then the unique Sylow 7-subgroup would be normal by the Sylow Theorems. This would contradict the fact that our group of order 168 is simple. The only remaining possibility is for n_7 to equal 8. This means that the number of elements of order 7 is 48.

Chapter 4.5 Exercise 36: Prove that if N is a normal subgroup of G then $n_p(G/N) \leq n_p(G)$.

Solution: Let S be the set of Sylow p-subgroups of G/N and let T be the set of Sylow p-subgroups of G. Let $\mathcal{P}(T)$ denote the power set of T, the set of all subsets of T (which includes the empty set and T itself). The goal for us is to construct a function

$$\mu : S \longrightarrow \mathcal{P}(T)$$

satisfying two conditions:
(i) for any $P \in S$, $\mu(P) \neq \emptyset$ and
(ii) if P and Q are two distinct Sylow p-subgroups of G/N, then $\mu(P) \cap \mu(Q) = \emptyset$. You should check immediately that $\mu(P) \cap \mu(Q) = \emptyset$ is equivalent to the statement that there are no Sylow p-subgroups of G that are subsets of $\pi^{-1}P \cap \pi^{-1}Q$.

Together, these two conditions imply that $|S| \leq |T|$ which is what we set out to prove.

Towards constructing μ, let $\pi : G \to G/N$ be the projection map. Given a Sylow p-subgroup $P \in G/N$, consider $\pi^{-1}P$ as a subgroup of G. Define $\mu(P)$ to be the set of Sylow p-subgroups of $\pi^{-1}P$. This requires explanation, since a priori, the Sylow p-subgroups of a subgroup of G are not necessarily Sylow p-subgroups of G. Now notice though that since P is a Sylow p-subgroup of G/N, $[G/N : P]$ is relatively prime to p. Also recall from an exercise many moons ago that $[G/N : P] = [G : \pi^{-1}P]$, so $[G : \pi^{-1}P]$ is also relatively prime to p. From the coset formula, we see then that $|G|$ and $|\pi^{-1}P|$ are divisible by the same power of p. Thus, any Sylow p-subgroup of $\pi^{-1}P$ is one of G also.

It still remains to check that the function μ satisfies the conditions (i) and (ii). (i) is satisfied by the existence portion of the Sylow Theorem which garauntees that for any P, there is at least one Sylow p-subgroup of $\pi^{-1}P$. As for (ii), suppose that $P \neq Q$ are two Sylow p-subgroups of G/N. Their intersection $P \cap Q$ must be a p-subgroup of G/N whose order is strictly less than $|P| = |Q|$. This forces $[G/N : P \cap Q]$ to be divisble by p. By the exercise from many moons ago, $[G/N : P \cap Q] = [G : \pi^{-1}(P \cap Q)]$ and a simple check shows that $\pi^{-1}(P \cap Q) = \pi^{-1}P \cap \pi^{-1}Q$. From these facts we deduce that $\pi^{-1}P \cap \pi^{-1}Q$ has index in G divisible by p. But then any subgroup of G contained in $\pi^{-1}P \cap \pi^{-1}Q$ cannot have order equal to the highest power of p dividing the order of G (i.e. $\pi^{-1}P \cap \pi^{-1}Q$ cannot contain any Sylow p-subgroups of G). This says precisely that $\mu P \cap \mu Q = \emptyset$.

Chapter 4.5 Exercise 40: Prove that the number of Sylow p-subgroups of $GL_2(\mathbb{F}_2)$ is $p+1$.

Solution: Okay, this one is a long one. Try to stick with it though. I'll break it into a couple parts.

Fact 1) $GL_2(\mathbb{F}_2)$ has $p(p-1)^2(p+1)$ elements.

We count them. The first column can be anything besides the zero vector. There are p^2-1 possibilities here. Now as long as the vector in the first column is nonzero, the second column can be any vector provided it is not a multiple of the first. Given a nonzero vector, there are p^2-p vectors which are not multiples of that vector. So the total number of matrices in $GL_2(\mathbb{F}_2)$ is $(p^2-1)(p^2-p) = (p+1)(p-1)(p-1)p = p(p-1)^2(p+1)$.

Fact 2) If G is a finite group, N is a normal subgroup of G, and P is a Sylow p-subgroup of G contained in N, then all the Sylow p-subgroups of G are contained in N and the Sylow p-subgroups of N are precisely the Sylow p-subgroups of G.

Let Q be any other Sylow p-subgroup. By the Sylow theorems, $Q = gPg^{-1}$ for some $g \in G$. But then we have $P \subset N \Rightarrow gPg^{-1} \subset gNg^{-1}$. Since N is normal, $gNg^{-1} = N$, so $Q = gPg^{-1} \subset N$. This settles the first claim. As for the second, since $P \subset N$, $|P|$ divides $|N|$. So the order of Sylow p-subgroups of N are the same as Sylow p-subgroups of G; thus any Sylow p-subgroups of H are Sylow p-subgroups of G, and any Sylow p-subgroups of G which happen to lie in H are Sylow p-subgroups of H. Since we've shown that all Sylow p-subgroups of G lie in H, the second claim follows.

Fact 3) There is a surjective group homomorphism $d : GL_2(\mathbb{F}_2) \to \mathbb{F}_p - \{0\}$.

This is just the determinant map defined in the usual way:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto ad - bc.$$

You may check that it is a group homomorphism. It is easy to see that it is surjective, since for any $x \in \mathbb{F}_p - \{0\}$, $d\left(\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}\right) = x$.

Okay, lets use these facts now. Let $SGL_2(\mathbb{F}_2)$ be defined as the kernel of the map d (i.e. those matrices whose determinant is 1). Since $SGL_2(\mathbb{F}_2)$ is the kernel of a group homomorphism it is a normal subgroup of $GL_2(\mathbb{F}_2)$. Using the formula

$$GL_2(\mathbb{F}_2)/SGL_2(\mathbb{F}_2) \approxeq \mathbb{F}_p - \{0\}$$

and the fact that all the groups are finite, we have $|SGL_2(\mathbb{F}_2)| = \frac{|GL_2(\mathbb{F}_2)|}{p-1} = \frac{p(p-1)^2(p+1)}{p-1} = (p-1)p(p+1)$.

Notice that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^l = \begin{bmatrix} 1 & l\text{mod}p \\ 0 & 1 \end{bmatrix}$. So the element $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ generates a group of order p (making it a Sylow p-subgroup of $GL_2(\mathbb{F}_2)$). Now also notice that the determinant of this matrix is 1, so this subgroup actually lies in $SGL_2(\mathbb{F}_2)$. Now we make use of Fact 2) to conclude that the Sylow p-subgroups of S/GL are the same as the Sylow p-subgroups of $GL_2(\mathbb{F}_2)$. Thus, to prove there are $p+1$ Sylow p-subgroups of $GL_2(\mathbb{F}_2)$ it suffices to show there are $p+1$ such subgroups of $SGL_2(\mathbb{F}_2)$.

The Sylow theorems tell us that the number of Sylow p-subgroups of $SGL_2(\mathbb{F}_2)$, n_p, is congruent to $1\text{mod}p$ and divides $(p-1)(p+1)$ (so $n_p c = (p-1)(p+1)$ for some positive integer c). It turns out that the only values n_p can take are 1 or $p+1$. To see this, first take the equation $n_p c = (p-1)(p+1)$ modulo p. It reduces to $c \equiv -1\text{mod}p$. Thus we have $n_p = kp+1$ and $c = lp-1$ where $0 < l$ and for the sake of contradiction we assume $2 \leq k$. We may also assume that k and l are both less than p since otherwise there is no way that $(kp+1)(lp-1) = p^2-1$. Expanding this equation out gives

$$\begin{aligned} klp^2 + (l-k)p - 1 &= p^2 - 1 \\ klp + (l-k) &= p \\ (kl-1)p = (l-k) & \end{aligned}$$

Since $k \geq 2$ and $l > 0$, $(kl-1) \neq 0$, we conclude that $l-k$ differ by a nonzero multiple of p. This is impossible since both l and k are less than p and nonnegative. To escape this contradiction, we must have either $k = 0$ or 1, i.e. $n_p = 1$ or $p+1$.

Finally, notice that just as the element $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ generated a Sylow p-subgroup of S/GL, so does the element $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ by an analogous argument.

It is apparent that these subgroups are different. We've constructed two Sylow p-subgroups, so there can't be just one. That leaves the only possible number of Sylow p-subgroups at $p+1$. Thanks for playing. Maybe there is a faster way. Let me know!

Chapter 4.6 Exercise 1: Prove that A_n does not have a proper subgroup of index $< n$ for all $n \geq 5$.

Solution: Suppose A_n DOES have a proper subgroup of index $k < n$, and that we call it H. After enumerating the left cosets of H, the action of A_n on these left cosets by left multiplication gives a group homomorphism

$$\phi : A_n \to S_k$$

The kernel of ϕ is a normal subgroup of A_n. Since $n \geq 5$, A_n is simple; $\ker\phi$ must be either the trivial group or A_n itself. Since A_n is a finite group, the group isomorphism $\frac{A_n}{\ker\phi} \approxeq \text{Im}\phi$ allows us to conclude that $|\ker\phi| = \frac{|A_n|}{|\text{Im}\phi|}$. But then we have

$$|\ker\phi| = \frac{|A_n|}{|\text{Im}\phi|} = \frac{\frac{n!}{2}}{|\text{Im}\phi|}.$$

Now $\text{Im}\phi \subset S_k$ implies $|\text{Im}\phi| \leq k!$. We plug this inequality into the formula above, giving us

$$|\ker\phi| \geq \frac{\frac{n!}{2}}{k!} \geq \frac{\frac{n!}{2}}{(n-1)!} > 1.$$

So the size of A_n and the size of S_k ensure that $\ker\phi$ cannot be the trivial group. On the otherhand, $\ker\phi$ acts trivially on all the cosets of H, and in particular it acts trivially on H itself. The only elements of A_n which act trivially on H by left multiplication are the elements of H itself, so $\ker\phi$ must be a subset of H. Since H was assumed to be a proper subgroup, this implies that $\ker\phi$ is a proper subgroup as well. We have reached a contradiction since $\ker\phi$ is a normal, nontrivial proper subgroup of A_n, which is a simple group.

Chapter 4.6 Exercise 2: Find all normal (nontrivial, proper) subgroups of S_n for all $n \geq 5$.

Solution: Let H be a normal nontrivial subgroup of S_n. Then $A_n \cap H$ is a normal subgroup of A_n (a two line exercise...do it!). Since $n \geq 5$, A_n is simple, so either $A_n \cap H$ is the trivial group, or $A_n \cap H$ is A_n. If the latter is true, then $A_n \leq H \leq S_n$. The index of A_n in S_n is two, and the index of H in S_n must then divide two. Well, there are only two ways this can happen. Either $[S_n : H] = 1$ in which case $H = S_n$ or $[S_n : H] = 2$ in which case $A_n = H$. Now suppose that $A_n \cap H = 1$. Since H is a normal subgroup we can use the isomorphism theorem

$$A_n H/H \approxeq A_n/A_n \cap H$$

which simplifies down by the assumption $A_n \cap H = 1$ to

$$A_n H/H \approxeq A_n$$

Since H was assumed to be nontrivial, and $A_n \cap H = 1$, we see that $A_n H$ must be strictly larger than A_n. By our previous discussion we know that the only subgroup of S_n containing A_n is S_n itself. So we must have

$$S_n/H \approxeq A_n$$

Since these groups are finite we see that $\frac{|S_n|}{|H|} = |A_n|$ which implies that $|H| = 2$.

Okay, lets summarize our results. We have found that if H is a normal, nontrivial, proper subgroup of S_n then either $H = A_n$ or $|H| = 2$. Suppose $|H| = 2$. The nonidentity element $x \in H$ must evidently has order two. Such an element must be a product of l disjoint transpositions where $1 \leq l \leq \lfloor \frac{n}{2} \rfloor$. Now for any such l, since $n > 2$, there is at least one other permutation y with the same cycle structure as x. Any two permutations with the same cycle structure are conjugate to one another, hence we can conjugate x to get y which is evidently not in H. Hence H is not normal. We've ruled out the possibility that $|H| = 2$. How about $H = A_n$? Well, A_n is normal. One reason for this is that any subgroup of index two is normal. More fancily done, we could use the fact that any subgroup whose index is the smallest prime dividing the order of the group is normal. Most obviously, we can just note that conjugation does not change the parity of a permutation.

In conclusion, the only normal, nontrivial, proper subgroup of S_n is A_n when $n \geq 5$.

Chapter 4.6 Exercise 3: Prove that A_n is the only proper subgroup of index $< n$ in S_n for all $n \geq 5$.

Solution: Suppose H is a proper subgroup of index $< n$ in S_n. We will show that $H = A_n$. Now the same argument used in **Chapter 4.6 Exercise 1** we may find a homomorphism $\phi : S_n \to S_{k<n}$ where $\ker\phi \leq H$ is a normal, nontrivial, proper subgroup of S_n. Since $n \geq 5$, we now use **Chapter 4.6 Exercise 2** to conclude that $\ker\phi = A_n$. Thus we have $A_n \leq H \leq S_n$. As H is proper, we must have $H = A_n$.

Chapter 5.4 Exercise 1: Prove that if $x, y \in G$ then $[y,x] = [x,y]^{-1}$. Deduce that for any subsets A and B of G, $[A,B] = [B,A]$ (recall that $[A,B]$ is the subgroup of G generated by the commutators $[a,b]$).

Solution:
(a) $[y,x][x,y] = (y^{-1}x^{-1}yx)(x^{-1}y^{-1}xy) = 1$. This shows that $[y,x] = [x,y]^{-1}$.
(b) Let S be subset of G, and let S^{-1} be the set of elements whose inverses are elements of S. Note that $(S^{-1})^{-1} = S$. Let $\langle S\rangle$ be the smallest subgroup containing S. Then it is clear that $\langle S\rangle = \langle S^{-1}\rangle$. In particular to this problem, let S be the set of commutators $[a,b]$ where $a \in A$ and $b \in B$ and let T be the set of commutators $[b,a]$. Part (a) of this problem shows that $S \subset T^{-1}$ and $T \subset S^{-1}$. Taking the inverse of all the elements in T and in S^{-1} we see that $T^{-1} \subset (S^{-1})^{-1} = S$. Hence $S \subset T^{-1}$ and $T^{-1} \subset S$, so $S = T^{-1}$. By our previous remarks we have

$$[A,B] := \langle S\rangle = \langle T^{-1}\rangle = \langle T\rangle := [B,A]$$

Chapter 5.4 Exercise 2: Prove that a subgroup H of G is normal if and only if $[G,H] \leq H$.

Solution: The subgroup $[G,H]$ is the smallest subgroup of G containing the commutators $[g,h]$ where $g \in G$ and $h \in H$. For this reason, $[G,H] \leq H$ is equivalent to the condition that every commutator $[g,h]$ is an element of H for every $g \in G$ and $h \in H$. But

$$g^{-1}h^{-1}gh \in H \;\Leftrightarrow\; g^{-1}h^{-1}g \in H \;\Leftrightarrow\; g^{-1}hg \in H$$

which is the definition of H being normal.

Chapter 5.4 Exercise 3: Let $a, b, c \in G$. Prove that
(a) $[a,bc] = [a,c](c^{-1}[a,b]c)$ (b) $[ab,c] = (b^{-1}[a,c]b)[b,c]$.

Solution:
(a) Plug and chug!

$$\begin{aligned}[a, bc] &= a^{-1}(bc)^{-1}a(bc) \\ &= a^{-1}c^{-1}b^{-1}abc \\ &= a^{-1}c^{-1}(ac)(ac)^{-1}b^{-1}abc \\ &= (a^{-1}c^{-1}ac)c^{-1}(a^{-1}b^{-1}ab)c \\ &= [a, c](c^{-1}[a, b]c).\end{aligned}$$

(b) Plug and chug!

$$\begin{aligned}[ab, c] &= (ab)^{-1}c^{-1}(ab)c \\ &= b^{-1}a^{-1}c^{-1}abc \\ &= b^{-1}a^{-1}c^{-1}(ac)(ac)^{-1}abc \\ &= b^{-1}(a^{-1}c^{-1}ac)c^{-1}a^{-1}abc \\ &= b^{-1}[a, c]c^{-1}bc \\ &= b^{-1}[a, c](bb^{-1})c^{-1}bc \\ &= b^{-1}[a, c]b(b^{-1}c^{-1}bc) \\ &= (b^{-1}[a, c]b)[b, c].\end{aligned}$$