HW 6 Solutions

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Chapter 4.3 Exercise 2: Find all conjugacy classes and their sizes in the following groups:
(a) D_8 (b) Q_8 (c) A_4

Solution:
(a) The conjugacy classes for D_8 are $\{1\}, \{r^2\}, \{r, r^3\}, \{s, sr^2\}, \{sr, sr^3\}$

(b) The conjugacy classes for Q_8 are $\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}$

(c) The conjugacy classes for A_4. If we were asking for the conjugacy for elements of A_4, *in* S_4 the answer would be easy, just given by cycle structure: $\{id\}, \{(12)(34), (13)(24), (14)(23)\}, \{(123), (213), (124), (214), (134), (314), (234), (324)\}$. But we are asked about conjugacy classes of these elements within A_4, so it is possible that some elements that were conjugate in S_4 are not conjugate in A_4.

In fact all the elements $\{(12)(34), (13)(24), (14)(23)\}$ are still conjugate in A_4. The key fact here is that for each such permutation, there is an odd permutation in its centralizer. For instance, (12) is in the centralizer of $(12)(34)$. Then we know that $(12)(34)$ must be conjugate to all other elements with that cycle structure. For instance if we know that there exists some $\sigma \in S_4$ such that $\sigma(12)(34)\sigma^{-1} = (14)(23)$. If σ is even, we're done. If we're not, we know that $(12)(12)(34)(12)^{-1} = (12)(34)$, so conjugating $(12)(34)$ by $\sigma(12)$ is also $(14)(23)$. But $\sigma(12)$ is an even permutation.

For the permutations of the cycle structure (123), the story is slightly different. We check by hand that the stabilizer of (123) consists of $\{id, (123), (123)^{-1} = (213)\}$. Thus, the number of elements in its conjugacy class is $\frac{|A_4|}{3} = 4$. Bumbling around for a short bit tells us that $\{(123), (142), (132), (124)\}$ is the conjugacy class containing (123). By a similar agument we find the the remaining conjugacy class is $\{(134), (234), (143), (243)\}$.

Chapter 4.3 Exercise 4: Prove that if $S \subset G$ and $g \in G$ then $gN_G(S)g^{-1} = N_G(gSg^{-1})$ and $gC_G(S)g^{-1} = C_G(gSg^{-1})$.

Solution:
(a) Let Φ_g denote conjugation by g, so $\Phi_g(x) := gxg^{-1}$. It is easy to check that for $g_1, g_2 \in G$ the equation $\Phi_g \Phi_h = \Phi_{gh}$ holds. Now suppose that h is an element of $N_G(gSg^{-1})$. So Φ_h restricts to a bijection on gSg^{-1}. Notice that Φ_g gives a bijection from S to gSg^{-1} and $\Phi_{g^{-1}}$ also gives a bijection from gSg^{-1} to S. Composing these bijections as follows

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "\Phi_g"'] & S \\
gSg^{-1} \arrow[r, "\Phi_h"'] & gSg^{-1} \arrow[u, "\Phi_{g^{-1}}"']
\end{tikzcd}$$

where the top map is the composition of the bottom three gives a bijection from S to S (it is a bijection since it is a composition of bijections). The top map is of course $\Phi_{g^{-1}} \Phi_h \Phi_g = \Phi_{g^{-1}hg}$. Thus $g^{-1}hg$ is an element of $N_G(S)$ which implies that h is an element of $gN_G(S)g^{-1}$. Hence $N_G(gSg^{-1}) \subset gN_G(S)g^{-1}$.

The opposite inclusion follows from the same argument as the first. Repeat the argument above but with gSg^{-1} in place of the set S, and the element g^{-1} in place of the group element g. Then the argument above proves that

$$N_G(g^{-1}(gSg^{-1})g) \subset g^{-1}N_G(gSg^{-1})g$$

which after simplifying becomes

$$N_G(S) \subset g^{-1}N_G(gSg^{-1})g$$

conjugate both sides by g. Doing so of course preserves the set inclusion:

$$gN_G(S)g^{-1} \subset gg^{-1}N_G(gSg^{-1})gg^{-1} = N_G(gSg^{-1}).$$

(b) The second problem can be solved in a similar manner. Keep the notation as before. If $h \in N_G(gSg^{-1})$ then we consider the series of maps

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "\Phi_g"'] & S \\
gSg^{-1} \arrow[r, "\Phi_h"'] & gSg^{-1} \arrow[u, "\Phi_{g^{-1}}"']
\end{tikzcd}$$

where the top map is the composition of the other maps. In this case though, the bottom map Φ_h restricts to the identity on gSg^{-1}. So we have $\Phi_{g^{-1}hg} = \Phi_{g^{-1}}\Phi_h\Phi_g = \Phi_{g^{-1}}id\Phi_g = \Phi_{g^{-1}}\Phi_g = \Phi_{g^{-1}g} = \Phi_1 = id$. Thus conjugation by $g^{-1}hg$ acts as the identity on S. So $g^{-1}hg \in C(S)$ which implies that $h \in gC(S)g^{-1}$. Hence $C_G(gSg^{-1}) \subset gC_G(S)g^{-1}$. Showing the opposite inclusion is done in exactly the same way as in part (a).

Chapter 4.3 Exercise 5: If the center, $Z(G)$ of G is of index n, prove that every conjugacy class has at most n elements.

Solution: If C is a conjugacy class and s is an element of C, recall that $|C|$ equals $[G : G_s]$. For any element g in the center of G we have $gsg^{-1} = gg^{-1}s = s$ because g commutes with all other elements of G. So $Z(G) \lhd G_s$. Using coset decomposition twice we see that

$$[G : Z(G)] = [G : G_s][G_s : Z(G)]$$

Since $[G : Z(G)] = n$ we must have $[G : G_s] \leq n$. So $|C| \leq n$.

Chapter 4.3 Exercise 7: For $n = 3, 4, 6$, and 7 make lists of the partitions of n and give representatives for the corresponding conjugacy classes of S_n.

Solution: This is just a rehash of **Proposition 13** and the ensuing example in Dummit & Foote. I'll just list the answers without explanation.

S_3:

Partition of 3	Representative of Conjugacy Class
1,1,1	1
1,2	(12)
3	(123)

S_4:

Partition of 4	Representative of Conjugacy Class
1,1,1,1	1
1,1,2	(12)
1,3	(123)
2,2	(12)(34)
4	(1234)

S_6:

Partition of 6	Representative of Conjugacy Class
1,1,1,1,1,1	1
1,1,1,1,2	(12)
1,1,1,3	(123)
1,1,4	(1234)
1,5	(12345)
1,1,2,2	(12)(34)
1,2,3	(12)(345)
2,2,2	(12)(34)(56)
2,4	(12)(3456)
3,3	(123)(456)
6	(123456)

S_7 :

Partition of 7	Representative of Conjugacy Class
1,1,1,1,1,1,1	1
1,1,1,1,1,2	(12)
1,1,1,1,3	(123)
1,1,1,2,2	(12)(34)
1,1,1,4	(1234)
1,1,5	(12345)
1,1,2,3	(12)(345)
1,6	(123456)
1,2,2,2	(12)(34)(56)
1,2,4	(12)(3456)
1,3,3	(123)(456)
2,2,3	(12)(34)(567)
2,5	(12)(34567)
3,4	(123)(4567)
7	(1234567)

Chapter 4.3 Exercise 9: Show that $|C_{S_n}((12)(34))| = 8\cdot(n-4)!$ for all $n \geq 4$. Determine the elements in this centralizer explicitly.

Solution: Recall that the centralizer of the element of (12)(34) is precisely the stabilizer of (12)(34) under the conjugation action. We have the formula $[G : G_{(12)(34)}] = |\text{conjugacy class represented by (12)(34)}|$. But we know the conjugacy class represented by (12)(34) consists of all pairs of disjoint transpositions. By counting we can see that the number of pairs of disjoint 2-tuples $(a_1a_2)(a_3a_4)$ where $1 \leq a_i \leq n$ is $n{\cdot}(n-1){\cdot}(n-2)(n-3)$. We also easily see that for every permutation represented by a pair of disjoint transpositions, there are 8 different disjoing $2-tuples$ that represent the same permutation. Hence the number of permutations in the conjugacy class of (12)(34) is $\frac{n\cdot(n-1)\cdot(n-2)(n-3)}{8}$. Then the size of the stabilizer, or centralizer of (12)(34) is $n!$ divided by this number. This yields $8{\cdot}(n-4)!$.

We give a description of the centralizer of (12)(34). Recall that the centralizer of (12)(34) are just those elements of S_n which commute with (12)(34). Disjoint permutations always commute, so any permutation which fixes $1, 2, 3$, and 4 is in the centralizer. This is a subgroup isomorphic to S_{n-4} after associating $5, 6, ..., n$ with $1, 2, ..., (n-4)$. Also you may check by hand that the elements (12), (34), and (14)(23) commute with (12)(34). You may check that these three elements generate a subgroup of order 8. Call this subgroup H. Then $H \times S_{n-4}$ is a subgroup of the stabilizer of (12)(34). Since $|H \times S_{n-4}| = 8{\cdot}(n-4)!$ evidently $H \times S_{n-4}$ is the centralizer of (12)(34).

Chapter 4.5 Exercise 3: Use Sylow's Theorem to prove Cauchy's Theorem (where only assume that G is finite with order divisible by p. We don't assume G is abelian).

Solution: Since p divides the order of G, we know by the Sylow's Theorem that there is a p-subgroup $P < G$. Let x be any nonzero element of P and consider the subgroup it generates. $|\langle x \rangle|$ is equal to the order of x. On the other hand, by Lagrange's theorem, $|\langle x \rangle|$ divides $|P|$. Since $|\langle x \rangle| \neq 1$, it must be itself a power of p. So say the order x is p^d for some $d > 1$. Then consider the element $x^{p^{d-1}}$. It clearly has order p.

Chapter 4.5 Exercise 6: Exhibit all Sylow 3-subgroups of A_4 and all Sylow 3-subgroups of S_4.

Solution: (a) Since $|A_4| = 12 = 2^2{\cdot}3$, this amounts to finding all subgroups of A_4 of order 3. From our work in the first question of this HW, it is clear that there are 4 of them, and they are generated by (123), (124), (134), and (234).

(b) The number of 3-Sylows is congruent to 1mod3 and divides 8. Hence there must be either 1 or 4 of them. Notice though that the 3-Sylows of S_4 have order 3, just like the 3-Sylows of A_4, so in fact all 3-Sylow subgroups of A_4 are Sylow 3-subgroups of S_4 as well. There can be at most 4 of them, and we've found 4, so we must have found them all.

Chapter 4.5 Exercise 13: Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing its order.

Solution: 56 factors as $2^3 \cdot 7$. By the Sylow Theorems, the number of 7-Sylow subgroups is $\equiv 1 \text{mod} 7$ and divides 8. Thus there are either 1 or 8 of these subgroups. If there is one of them, then by the Sylow Theorem says that it is normal. Otherwise, there are 8 subgroups isomorphic to $\mathbb{Z}/7$. If this were the case, then since the intersection of any two of these subgroups is just the singleton set containing the identity (Lagrange's theorem), the number of elements of order 7 is 48. If this is the case, then that leaves 8 elements which could belong to a 2-Sylow subgroup. Any Sylow 2-subgroup has 8 elements, so evidently, there can only be one Sylow 2-subgroup. By Sylow's theorem, this subgroup would be normal.

Chapter 4.5 Exercise 14: Prove that a group of order 312 has a normal Sylow p-subgroup for some prime p dividing its order.

Solution: 312 factors as $2^3 \cdot 3 \cdot 13$. The number of Sylow 13-subgroups is $\equiv 1 \text{mod} 13$ and divides 24. The only positive integer with this property is 1. Thus, there is precisely one Sylow 13-subgroup, hence by the Sylow Theorems it must be normal.