Mirror Symmetry: Introduction to the B Model

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1 Introduction

Recall that mirror symmetry predicts the existence of pairs $X, \tilde{X}$ of Calabi-Yau manifolds whose Hodge diamonds are mirror images of each other, i.e. $H^q(X, \Lambda^pTX) \simeq H^q(\tilde{X}, \Omega^p\tilde{X})$. In fact, mirror symmetry reflects much more than just the Hodge structures, and we also get an isomorphism between the “Yukawa couplings” on $H^1(X,TX)$ and $H^1(\tilde{X})$, which give product structures $H^1(X,TX) \otimes H^1(X,TX) \to H^1(X,TX)$ and $H^{1,1}(\tilde{X}) \otimes H^{1,1}(\tilde{X}) \to H^{1,1}(\tilde{X})$. The coupling on $H^1(X,TX)$ is defined in terms of Gromov-Witten invariants of $X$ and often contains deep enumerative information about $X$. For example, when $X$ is the quintic threefold, it is much easier to calculate the Yukawa couplings for the mirror $\tilde{X}$, and mirror symmetry then gives astonishing formulas for the number of degree $d$ rational curves on $X$, for all $d$.

We aim to give a more refined statement of mirror symmetry in terms of the complex and Kahler moduli spaces of $X$ and $\tilde{X}$ respectively. Let $\mathcal{M}_{\text{cx}}(X)$ and $\mathcal{M}_{\text{kah}}(\tilde{X})$ denote the complex and Kahler moduli spaces of $X$ and $\tilde{X}$ respectively. The former is defined to be the moduli space of complex structures on $X$. Assuming $h^{2,0}(\tilde{X}) = 0$, the latter can be defined in terms of the Kahler cone $K(\tilde{X}) \subset H^2(\tilde{X}, \mathbb{R})$ consisting of all Kahler classes. Namely, we define $\mathcal{M}_{\text{kah}}(\tilde{X})$ to be $K_{\mathbb{C}}(\tilde{V})/\text{Aut}(\tilde{X})$, where $K_{\mathbb{C}}(\tilde{X})$ is the “complexified Kahler space” given by

$$K_{\mathbb{C}}(\tilde{X}) = \{\omega \in H^2(\tilde{X}, \mathbb{C}) | \text{Im}(\omega) \in K(\tilde{X})\}/\text{im}H^2(\tilde{X}, \mathbb{Z}).$$

The goal of this talk is to discuss the following statement of mirror symmetry:

**Conjecture 1.1** Let $\mathcal{X} \to (D^*)^s$ be a family of Calabi-Yau 3-folds with a large complex structure limit (LCSL) point at 0. Then there is another Calabi-Yau 3-fold $\tilde{X}$ and a choice of bases

$$\alpha_0, \ldots, \alpha_s, \beta_0, \ldots, \beta_s \quad \text{for} \quad H_3(X, \mathbb{Z})$$

$$e_1, \ldots, e_3 \quad \text{on} \quad H^2(\tilde{X}, \mathbb{Z})$$

1
giving rise to a locally defined map

\[ m : \mathcal{M}_{\text{cx}}(X) \to \mathcal{M}_{\text{kah}}(\check{X}), \]

\[(q_1, ..., q_s) \mapsto (\check{q}_1, ..., \check{q}_s),\]

such that the Yukawa couplings match:

\[ \left< \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k} \right>_{\mu} = \left< \frac{\partial}{\partial \check{q}_i}, \frac{\partial}{\partial \check{q}_j}, \frac{\partial}{\partial \check{q}_k} \right>_{m(\mu)}. \]

Here the LCSL condition essentially corresponds to a complexified Kahler form \([B + i\omega]\) for \(\omega\) sufficiently positive. We will show that the bases give rise to local coordinates \(q_i\) and \(\check{q}_i\) on \(\mathcal{M}_{\text{cx}}(X)\) and \(\mathcal{M}_{\text{kah}}(\check{X})\) and thus the map \(m\).

**Example 1.2** Consider the family of elliptic curves

\[ C_t = \{ y^2z = x^3 + x^2z - tz^3 \} \subset \mathbb{CP}^2. \]

Note that \(C_t\) is smooth for \(t \neq 0\), and \(C_0\) has a nodal singularity. As \(t\) travels around the origin, the monodromy is a Dehn twist around a vanishing cycle. The induced map on homology \(H_1(C_{t_0}) \simeq \mathbb{Z}^2 = \langle a, b \rangle\) is given by \(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\).

Using period integrals, we replace the ad hoc parameter \(t\) with a natural local coordinate \(q\) for the family. First, equip each \(C_t\) with a holomorphic volume form \(\Omega_t\) such that, for each \(t\),

\[ \int_a \Omega_t = 1. \]

Now let

\[ \tau(t) = \int_b \Omega_t. \]

Abstractly, we have \(C_t \simeq \mathbb{C}/(\mathbb{Z} + \tau(t)\mathbb{Z})\). As \(t\) goes around the origin, \(\tau(t)\) goes to \(\tau(t) + \int_a \Omega_{t_0} = \tau(t) + 1\). Therefore \(q(t) := e^{2\pi i \tau(t)}\) is single-valued and gives a local coordinate for the family. As \(t \to 0\), \(\text{Im } \tau(t) \to \infty\) and \(q(t) \to 0\). This is an example of a LCSL.

## 2 Deformations of Complex Structures

For a given Calabi-Yau manifold, we would like to study the local structure of the moduli space of complex structures. To start, let \((X, J)\) be an almost complex manifold. Recall that we have a decomposition

\[ TX \otimes \mathbb{C} = TX^{1,0}_J \oplus TX^{0,1}_J \]
of the complexified tangent bundle of $X$ into the $i$ and $-i$ eigenspaces respectively of $J$. Note that for $v \in TX \otimes \mathbb{C}$, we have $v = v_{j,0}^{1} + v_{j,1}^{0}$, with $v_{j,0}^{1} = \frac{1}{2}(v - iJv) \in TX_{j}^{1,0}$ and $v_{j,1}^{0} = \frac{1}{2}(v + iJv) \in TX_{j}^{0,1}$. Similarly, we have decompositions

$$T^*X \otimes \mathbb{C} = T^*X_{j}^{1,0} \oplus T^*X_{j}^{0,1},$$

$$\Lambda^k T^*X = \oplus_{p+q=k} \Omega^p q_j(X).$$

Now for $J'$ another almost complex structure close to $J$, we can view $\Omega_{j'}^{1,0} \subset T^*X \otimes \mathbb{C} = \Omega_j^{1,0} \oplus \Omega_j^{0,1}$ as the graph of a linear map $s : \Omega_{j}^{1,0} \to \Omega_{j}^{0,1}$. Conversely, for such an $s$ sufficiently small we can set

$$\Omega_{j'}^{1,0} := \text{graph}(s),$$

$$\Omega_{j'}^{0,1} := \Omega_{j}^{1,0},$$

and then define the action of $J'$ to be multiplication by $i$ on $\Omega_{j'}^{1,0}$ and by $-i$ on $\Omega_{j'}^{0,1}$. Note that $s$ can also be viewed as a section of $(\Omega_{j}^{1,0})^* \otimes \Omega_{j}^{0,1} \simeq T_{j}^{1,0} \otimes \Omega_{j}^{0,1}$.

Of course, we want the deformation $J'$ to be integrable. Recall that the almost complex structure $J$ is integrable if and only if we have

$$[TX_{1,0}, TX_{1,0}] \subset TX_{1,0}.$$ 

Note that the Dolbeaut complex for $TX_{j}^{1,0}$ on $(X, J)$, namely $\oplus_q \Omega^q X \otimes TX_{j}^{1,0}$, carries a Lie bracket given by

$$[\alpha \otimes v, \alpha' \otimes v'] := (\alpha \wedge \alpha') \otimes [v, v'].$$

Using local coordinates, one can easily show:

**Proposition 2.1** $J'$ is integrable if and only if $\partial s + \frac{1}{2}[s, s] = 0$.

Recall that we also need to quotient by Diff($X$). Let $\phi$ be a diffeomorphism of $X$ which is close to the identity. We first remark that for $\{z_i\}$ local holomorphic coordinates for $(X, J)$, one can check that a basis for the $(1, 0)$-forms corresponding to $J'$ is given by

$$\{dz_i - s(dz_i)\}.$$ 

Decomposing $d\phi$ into parts which commute and anticommute with $J$:

$$d\phi = \partial \phi + \bar{\partial} \phi,$$
we therefore have
\[ \phi^* dz_i = dz_i \circ \partial \phi + dz_i \circ \overline{\partial} \phi = (dz_i \circ \partial \phi) \circ (\text{Id} + (\partial \phi)^{-1} \overline{\partial} \phi), \]
and hence corresponding to \( \phi^* J \) we have
\[ s = -(\partial \phi)^{-1} \overline{\partial} \phi. \]

Putting this all together, consider a deformation \( J(t) \) of \( J \) with \( J(0) = J \), corresponding to \( s(t) = ts_1 + t^2 s_2 + t^3 s_3 + \ldots \in \Omega^{0,1}(X, TX^{1,0}) \). Assuming the family \( J(t) \) is integrable, the equation
\[ \overline{\partial} s(t) + \frac{1}{2} \{ s(t), s(t) \} = 0 \]
gives
\[ \frac{1}{2} \{ ts_1 + t^2 s_2 + \ldots, ts_1 + t^2 s_2 + \ldots \} + t \overline{\partial} s_1 + t^2 \overline{\partial} s_2 + \ldots = 0. \]
In particular, the first order part in \( t \) gives
\[ \overline{\partial} s_1 = 0. \]

On the other hand, if \( \phi_t \) is a family of diffeomorphisms of \( X \) with \( \phi_0 = \text{Id} \), we have
\[ \left. \frac{d}{dt} \right|_{t=0} \left( - (\partial \phi_t)^{-1} \overline{\partial} \phi_t \right) = - \left. \frac{d}{dt} \right|_{t=0} (\overline{\partial} \phi_t) = - \overline{\partial} v, \]
where \( v \in \Gamma(TX) \) is the vector field generating the family \( \phi_t \). In other words, first order deformations of \( (X, J) \) correspond to
\[ \text{Ker}(\overline{\partial} : \Omega^{0,1}(X, TX^{1,0}) \to \Omega^{0,2}) \]
\[ \cong \text{Im}(\overline{\partial} : C^\infty(X, TX^{1,0}) \to \Omega^{0,1}) = H^1(X, TX^{1,0}). \]
Moreover, if \( (X^n, J) \) is Calabi-Yau with \( \Omega \) a holomorphic volume form, then \( \Omega \) gives an identification \( TX^{1,0} \cong \wedge^{n-1} T^* X \), and therefore we have
\[ H^1(X, TX^{1,0}) \cong H^1(X, \wedge^{n-1} T^* X) \cong H^{n-1,1}(X, J). \]

A priori there may be obstructions to finding the higher order parts of \( s(t) \). Namely, we must have
\[ \overline{\partial} s_2 + \frac{1}{2} \{ s_1, s_2 \} = 0, \]
\[ \overline{\partial} s_3 + [s_1, s_2] = 0, \]
\[ \overline{\partial} s_4 + [s_1, s_3] + \frac{1}{2} [s_2, s_2] = 0, \]
etc, and so there are obstructions lying in \( H^2(X, TX) \). Happily, we have

**Theorem 2.2 (Bogomolov-Tian-Todorov)** For \( X \) a compact Calabi-Yau with \( H^0(X, TX) = 0 \), deformations of \( X \) are unobstructed.
3 The Hodge Bundle

Partly for convenience, we now focus on three dimensional Calabi-Yau manifolds. In this case, the cohomology groups $H^3(X; \mathbb{C})$ glue together to form a bundle $\mathcal{H}$, the Hodge bundle, over the moduli space of complex structures. The Calabi-Yau forms, unique up a constant, form a line sub-bundle of the Hodge bundle. Moreover, by declaring integer cohomology classes to be flat sections, we get a connection the Hodge bundle, the Gauss-Manin connection.

It turns out that the position of the Calabi-Yau form (up to scaling) in $H^3$ (locally) determines the complex structure. We can think of the Calabi-Yau form as determining a point in $\mathbb{P}^{h^3-1}$, where $h^3$ is the dimension of $H^3$. Since the dimension of $\mathcal{M}_{\text{cx}}(X)$ is only $h^{2,1} = \frac{1}{2} h^3 - 1$, this description of the complex structure is redundant. To find sharper coordinates on $\mathcal{M}_{\text{cx}}(X)$, we first define a natural Hermitian metric $(\cdot, \cdot)$ on $H^3$ by

$$(\theta, \eta) = i \int \theta \wedge \overline{\eta}, \quad \theta, \eta \in H^3(M, \mathbb{C}).$$

Then we can find a “symplectic basis” of real integer three-forms $\alpha_a, \beta^b$, $a, b = 1, \ldots, h^3/2$, such that

$$(\alpha_a, \alpha_b) = (\beta^a, \beta^b) = 0$$

$$(\alpha_a, \beta^b) = i \delta^b_a,$$

with dual basis $A^a, B_b$, $a, b = 1, \ldots, h^3/2$.

4 Periods and Coordinates on Moduli Space

We first discuss the coordinates on $\mathcal{M}_{\text{kah}}(\hat{X})$. If $e_i$ is a basis for $H^2(\hat{X}, \mathbb{Z})$ with each $e_i$ in the Kahler cone, we get local coordinates $\tilde{q}_i$ on $\mathcal{M}_{\text{kah}}(\hat{X})$ by setting

$$[B + i\omega] = \sum_i \tilde{t}_i e_i,$$

$$\tilde{q}_i = \exp(2\pi i \tilde{t}_i) \in \mathbb{C}^*.$$

Now to construct coordinates on $\mathcal{M}_{\text{cp}}(X)$, let $\Omega$ be a Calabi-Yau form. We consider the “period integrals”

$$z^a = \int_{A^a} \Omega, \quad \omega_b = \int_{B^b} \Omega.$$

It turns out that the complex structure is locally determined by just the $z^a$. In fact, there are $h^3/2$ of them, while the moduli space is only of dimension $h^{2,1} = h^3/2 - 1$. However, the Calabi-Yau form $\Omega$ is only well-defined up to scaling, so we view the $z^a$ as homogenous local coordinates.
Now suppose $X \to (D^*)^s$ is a LCSL. If $h^{n-1,1} = s = 1$, this equivalent to the monodromy $\phi_\ast \in \text{Aut}(H^n(X_{t_0}, \mathbb{Z}))$ around 0 being \textit{maximally unipotent}, i.e.

$$(\phi_\ast - \text{Id})^k = 0$$

for $k = n + 1$ but not for $k < n + 1$. For $s > 1$ there is a more involved definition in terms of the Jordan decompositions of the various induced monodromy actions on $H^n(X_{t_0}, \mathbb{Z})$ corresponding to loops in $(D^*)^s$. Let $\Omega$ be such that $\int_{B_0} \Omega = 1$. Then setting $q_i = \exp(2\pi i \omega_i)$ defined canonical coordinates $q_i$ on $(D^*)^s$ (which of course are only canonical after a basis is chosen).

5 \textbf{Yukawa Couplings}

On the $A$ side, the Yukawa couplings are given as follows. For $\omega_1, \omega_2, \omega_3 \in H^{1,1}(\check{X})$,

$$\langle \omega_1, \omega_2, \omega_3 \rangle := \int_X \omega_1 \wedge \omega_2 \wedge \omega_3 + \sum_{0 \neq \beta \in H^2(X, \mathbb{Z})} n_\beta \int_\beta \omega_1 \int_\beta \omega_2 \int_\beta \omega_3 \frac{e^{2\pi i f_\beta \omega}}{1 - e^{2\pi i f_\beta \omega}},$$

where $n_\beta$ is defined in terms of Gromov-Witten invariants and is roughly the “number of holomorphic spheres in $\check{X}$ of class $\beta$”. The numbers $n_\beta$ contain deep information about the arithmentic properties of $\check{X}$.

On the $B$ side, the Yukawa coupling for $\theta_1, \theta_2, \theta_3 \in H^1(X, T_X)$ is given by

$$\langle \theta_1, \theta_2, \theta_3 \rangle := \int_X \Omega \wedge (\theta_1 \cdot \theta_2 \cdot \theta_3 \cdot \Omega),$$

using the composition

$$S^3 H^1(X, T_X) \otimes H^0(X, \Omega^3 X) \to H^3(X, \Lambda^3 T_X \otimes \Omega^3 X) \simeq H^3(X, \mathcal{O}_X) \simeq H^{0,3}(X).$$

Equivalently, this is

$$\int_X \Omega \wedge (\nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega),$$

where $\nabla$ is the Gauss-Manin connection.

\textbf{Remark 5.1} Using $z^a$ and $\omega_b$, we define the “prepotential” $\mathcal{G}$:

$$\mathcal{G} := z^a \omega_a.$$

Then using $\mathcal{G}$ we can recover the Yukawa couplings

$$\kappa_{a,b,c} = \langle \chi_a, \chi_b, \chi_c \rangle,$$

where $\chi_a$ is the $(2,1)$ part of $\partial_\ast \Omega$ (considered as an element of $H^1(TM)$, by

$$\kappa_{a,b,c} = \partial_a \partial_b \partial_c \mathcal{G}.,$$
6 Mirror Symmetry for the Quintic Threefold

Let \( V \subset \mathbb{CP}^4 \) be a smooth quintic hypersurface, i.e. the zero set of a homogeneous degree 5 polynomial. Then \( V \) is Calabi-Yau. If \( H \) denotes the hyperplane class, the \( A \)-model Yukawa coupling formula becomes

\[
\langle H, H, H \rangle = 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1 - q^d}
\]

where \( q = \exp(2\pi i \int \omega) \), \( l \) is a line in \( V \), and \( \omega = B + iJ \) is a complexified Kahler class on \( V \). The first few values of \( n_d \) are given by

\[
\begin{align*}
    n_1 &= 2,875 \\
n_2 &= 609,250 \\
n_3 &= 317,206,375 \\
n_4 &= 242,467,530,000,
\end{align*}
\]

etc.

**Remark 6.1** Although \( n_d \) is indeed the number of rational curves of degree \( d \) in \( V \) for \( d \leq 9 \), in general the enumerative content of \( n_d \) is more subtle. In particular, \( n_{10} \) does not give the number of degree 10 rational curves on \( V \), as double covers of nodal rational curves contribute more than expected.

One of the most striking early applications of mirror symmetry to mathematics was the computation of the above expression using the \( B \)-model couplings on the mirror of \( V \). In this last section we explain the mirror of \( V \) and how to get the mirror map.

Recall that the Hodge diamond of \( V \) is given by \( h^{1,1}(V) = 1 \) and \( h^{2,1}(V) = 101 \). Therefore the mirror \( \tilde{V} \) should satisfy \( h^{2,1}(\tilde{V}) = 1 \) and hence live in a one-parameter family of Calabi-Yau manifolds. We describe it as a resolution of singularities of a family of hypersurfaces in \( \mathbb{CP}^4/G \), where

\[
G = \{ (a_1, ..., a_5) \in (\mathbb{Z}/5)^5 \mid \sum_i a_i \equiv 0 \mod 5 \}/(\mathbb{Z}/5),
\]

and \( g = (a_1, ..., a_5) \) acts by

\[
g \cdot (x_1, ..., x_5) = (\mu^{a_1} x_1, ..., \mu^{a_5} x_5)
\]

for \( \mu = \exp(2\pi i / 5) \).

With parameter \( \psi \), the hypersurfaces are defined by

\[
x_1^5 + ... + x_5^5 + \psi x_1 x_2 x_3 x_4 x_5 = 0.
\]
These hypersurfaces inherit singularties from $\mathbb{CP}^4/G$, and we let $\tilde{V}_\psi$ be the result after simultaneously resolving the singularities. Now we observe that the map

$$(x_1, ..., x_5) \mapsto (\mu^{-1}x_1, x_2, ..., x_5)$$

induces an isomorphism $\tilde{V}_\psi \simeq \tilde{V}_{\mu\psi}$, and $\psi^5$ is well-defined on the moduli space of complex structures on $\tilde{V}_\psi$. We set $x = \psi^{-5}$ as a local coordinate for the complex moduli. The singularities of $\tilde{V}_\psi$ occur for $\psi = -5\mu^i$, $0 \leq i \leq 4$, and for $\psi = \infty$. In terms of $x$, they occur for $x = -5^{-5}$ and $x = 0$.

Now since $H$ generates $H^2(V, \mathbb{Z})$, we can write any complexified Kahler class as $\omega = tH$ for $t$ in the upper half plane. Since $K^*_C(V)$ is the quotient of $K_C(V)$ by $H^2(V, \mathbb{Z})$, setting $q = \exp(2\pi it)$ gives an isomorphism

$$q : K^*_C(V) \simeq \Delta^*.$$ 

It turns out that the limit point $0 \in \Delta$ corresponds to a LCSL. In the complex structure moduli space, the point $x = 0$ has maximally unipotent monodromy and therefore should be the image of $q = 0$.

However, the mirror map $\mathcal{M}_{kah}(V) \to \mathcal{M}_{cx}(\tilde{V})$ is not given by $q = x$. Rather, we put a coordinate $\tilde{q}$ on $\tilde{V}$ as follows. We claim that there is a minimal integral vanishing cycle $\gamma_0$ near $x = 0$ such that $\gamma_0$ is invariant under monodromy, and that there is a minimal integral cycle $\gamma_1$ which transforms under the monodromy about $x = 0$ by $\gamma_1 \mapsto \gamma_1 + \gamma_0$. Then for $\Omega$ a holomorphic 3-form, monodromy around $x = 0$ gives the transformation

$$\int_{\gamma_1} \Omega / \int_{\gamma_0} \Omega \mapsto \int_{\gamma_1} \Omega / \int_{\gamma_0} \Omega + 1,$$

and we set

$$\tilde{q} = \exp(2\pi \int_{\gamma_1} \Omega / \int_{\gamma_0} \Omega).$$

Then $q$ and $\tilde{q}$ correspond under the mirror map.

**Remark 6.2** In order to equate the Yukawa couplings, the next step would be to find an expression for $\tilde{q}$ in terms of $x$. To do this, we can use the fact that $\int_{\gamma_0} \Omega$ and $\int_{\gamma_1} \Omega$ are periods and therefore satisfy a Picard-Fuchs equation of the form

$$y''' + f_1y'' + f_2y' + f_3y + f_4y = 0,$$

for the $f_i$ functions of $x$ and differentiation taken with respect to $x$. This comes from the fact that $h^3(\tilde{V}) = 4$ and hence any 5 sections of the Hodge bundle must be linearly dependent. Using techniques from ordinary differential equations, we ultimately find that

$$\tilde{q} = -(x - 770x^2 + \ldots).$$
References
