1 Finding extrema of functions on a domain with boundary

Example 1.1 Let $S$ be the semidisc in $\mathbb{R}^2$ defined by the inequalities

$$x \geq 0 \quad y \leq \sqrt{1-x^2}.$$ 

If $f : S \to \mathbb{R}$ is defined by $f(x, y) = xy$, find the maximum value of $f$.

2 Finding local and global extrema of functions

Example 2.1 Find all critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 5 + (x - 1)(y - 3)$ and classify them as local minima, local maxima, or saddle points.

Example 2.2 Find all critical points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 4 + x^3 + y^3 - 3xy$ and classify them as local minima, local maxima, or saddle points.

Example 2.3 Determine the point on the plane $4x - 2y + z = 1$ that is closest to the point $(-2, -1, 5)$.

3 Exercises

Exercise 3.1 (Final Exam, Winter 2013)

Let $f(x, y) = 3x^2y + y^3 + 6xy$.

(a) Find all the critical points of $f$. (There are four, all with integer coordinates).

(b) Classify each critical point of $f$ as a local minimum, local maximum, or saddle point.

(c) Show that $f$ does not have any absolute extrema on $\mathbb{R}^2$. (This does not require the previous parts).
Exercise 3.2  (Final Exam, Spring 2012) Find the maximum and minimum values of $f(x, y) = 4y^2 + x^2 - 6x$ on the region

$$D = \{(x, y) : x^2 + y^2 \leq 4\}.$$ 

Exercise 3.3  (Final Exam, Spring 2011) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by the formula

$$f(x, y) = 2x^3 + 2y^3 + 9x^2 - 3y^2 - 12y.$$ 

(a) Show that the only critical points of $f$ are $(0, -1), (0, 2), (-3, -1), (-3, 2)$.

(b) Use the Second Derivative Test to characterize each of the critical points $(-3, -1)$ and $(0, -1)$ as a local maximum, local minimum, or neither.

(c) Does $f$ have a global minimum value on $\mathbb{R}^2$? Justify fully.

Exercise 3.4 Suppose $f(x, y) = 2x^2 + xy - 8x - y + 6$. Let $T \subset \mathbb{R}^2$ be the region enclosed by the triangle whose vertices are $(0, 0), (0, 3), (3, 0)$. (Points on the triangle itself are also taken to lie in $T$.) Find, with complete justification, the absolute extreme values of $f$ on $T$. 

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