1 Directional Derivatives and Gradients

Example 1.1 Let \( f(x,y,z) = x^2yz - xz^2 + 3y^2 \), \( \mathbf{v} = \begin{pmatrix} 5 \\ 0 \\ -12 \end{pmatrix} \), \( \mathbf{a} = (-2,-1,1) \). Find the directional derivative \( D_{\mathbf{v}}f(\mathbf{a}) \) in two ways: first by computing the gradient, and then by parameterizing a line in \( \mathbb{R}^3 \) passing through \( \mathbf{a} \) with velocity \( \mathbf{v} \), and composing this with the function \( f \).

Example 1.2 A crow is at the point \( (3,-1,6) \) in a region of the sky where the humidity is given by

\[
h(x,y,z) = (x + 2y + z)^2 - (y + z)^2 + 3z^2.
\]

If it flies in the direction \( \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \), is the humidity increasing or decreasing?

It is a very useful fact that if \( S \) is a level surface given by \( S = F^{-1}(C) \), for \( C \in \mathbb{R} \) and \( F : \mathbb{R}^3 \to \mathbb{R} \) a function, then if \( a \) is a point on \( S \), the gradient of \( F \) at \( a \) points in a direction orthogonal to \( S \) at that point\(^1\). This makes sense if you think of the gradient as telling you the direction of steepest increase. Try to utilize this in the following problem.

Example 1.3 Consider the paraboloid in \( \mathbb{R}^3 \) given by the equation

\[
y = x^2 + z^2.
\]

Find the tangent plane to this paraboloid at the point \( (1,1,0) \).

Exercise 1.4 (Midterm 2, Autumn ’08) Find the equation of the tangent plane to the surface \( z = x^2y + y^3 \) at the point \( (2,1,5) \).

\(^1\)There are some mild technical assumptions that we also need but let’s ignore those for now.
2 More Problems

Exercise 2.1 (Midterm 2, Autumn ’10) For each function, choose one of the following statements:

- Each non-empty level set is a line (L)
- Each non-empty level set is a circle (C)
- Not all the level sets are lines or circles (N)

1. \( r(x, y) = e^{2x+y} \)
2. \( g(x, y) = \sin^2(xy) + \cos^2(xy) \)
3. \( h(x, y) = \sin(xy) \)
4. \( s(x, y) = (x + y)^2 + (x - y)^2 \)
5. \( f(x, y) = x + (\ln \pi)y \)

Exercise 2.2 (Midterm 2, Autumn ’9) Let \( Q(x, y) = x^2 - 2axy + y^2 \). For what values of \( a \) is the quadratic form \( Q \) negative definite?

Exercise 2.3

\[ \lim_{(x,y) \to (\pi,e)} \frac{(x - \pi)^2 + 3(x - \pi)(y - e) + (y - e)^2}{(x - \pi)^2 + (y - e)^2} \]

Exercise 2.4

\[ \lim_{(x,y) \to (0,0)} \frac{x^2(x + y)}{x^2 + y^2} \]

Exercise 2.5 Consider the helix, which is the image of the parameterized curve \( f(t) = (\cos t, \sin t, t) \). Find the plane \( P \) in \( \mathbb{R}^3 \) which passes through the point \( p_0 = (1,0,4\pi) \) and which intersects the helix orthogonally at that point.