1 Midterm Review Exercises

Exercise 1.1 (Midterm 2, Winter ’11)
(a) Let
\[ f(x, y) = \frac{x^2 y + xy^2 + y^3}{x^2 + y^2}. \]
Find
\[ \lim_{(x,y) \to (0,0)} f(x,y) \]
if the limit exists.
(b) Compute \( \frac{\partial f}{\partial x} \), and use this to determine
\[ \lim_{(x,y) \to (0,0)} \frac{\partial f}{\partial x} \]
if the limit exists.

Exercise 1.2 (Midterm 2, Winter ’11) Let \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) be the function
\[ g(x, y) = (\sin(x + 3y), xy^2 + y) \]
and suppose that \( f \) is a function defined on a neighborhood of \( (0,0) \), such that the composition \( f \circ g \) is the identity function. Find \( D_f(0,0) \).

Exercise 1.3 (Midterm 2, Autumn ’12) Define a transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) by first rotating counterclockwise by \( \pi/2 \) and then multiplying by the matrix \( A = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \).
(a) Find a matrix \( B \) so that \( T(x) = Bx \) for all \( x \in \mathbb{R}^2 \).
(b) What is the area of the image of the unit square (vertices \( (0,0), (1,0), (1,1), (0,1) \)) under \( T \)?
(c) How would your answers to (a) and (b) change if you first multiplied by \( A \) and then rotated by \( \pi/2 \).
Exercise 1.4 (Midterm 2, Autumn ’12) Let \( A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \).

(a) Compute the determinant of \( A \).
(b) Find the inverse of \( A \).
(c) Let \( A \) be the \( 3 \times 3 \) matrix given in part (a). Suppose \( B \) is another \( 3 \times 3 \) matrix such that
\[
AB = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 3 & 0 \end{pmatrix}.
\]
What is \( B \)?

Exercise 1.5 (Midterm 2, Autumn ’12) Let \( A \) be the matrix \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \), and let \( Q_A : \mathbb{R}^2 \to \mathbb{R} \) be the associated quadratic function
\[
Q_A(x) = x^T Ax.
\]

(a) Find \( \frac{\partial Q_A}{\partial x} \) and \( \frac{\partial Q_A}{\partial y} \).
(b) Determine whether \( Q_A \) is positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite.

Exercise 1.6 (Midterm 2, Autumn ’12) True or False.

1. Let \( v \) be a nonzero vector in \( \mathbb{R}^3 \), and let \( w \) be another vector which is not a multiple of \( v \). Then the \( 3 \times 3 \) matrix whose columns are the three vectors, \( \{v, w, v \times w\} \) has nonzero determinant.

2. If \( A \) and \( B \) are square matrices, \( C = AB \), then \( \operatorname{rank}(A) \geq \operatorname{rank}(C) \).

3. Let \( A \) be any \( 2 \times 2 \) matrix whose determinant is nonzero. Then \( A \) has at least one (real) eigenvalue.

4. Suppose \( A, B, \) and \( C \) are \( n \times n \) matrices such that \( A = C^{-1}BC \). Then \( \det(A) = \det(B) \).

Exercise 1.7 (Midterm 2, Autumn ’12) Consider the following basis for \( \mathbb{R}^2 \):
\[
\mathcal{B} = \{v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}\}.
\]
Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation that has the matrix \( B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \) with respect to the basis \( \mathcal{B} \). Find the matrix \( A \) for \( T \) with respect to the standard basis for \( \mathbb{R}^2 \).
Exercise 1.8 (Midterm 2, Autumn ’12)
(a) Let $M = \begin{pmatrix} 0 & 4 \\ 9 & 0 \end{pmatrix}$. Find the eigenvalues of $M$, and bases of the corresponding eigenspaces.

(b) Let $M$ be the $2 \times 2$ matrix given in part (a). Find a diagonal matrix $D$ and a matrix $C$ such that $M = CDC^{-1}$.

Exercise 1.9 (Midterm 2, Autumn ’12) Let $P$ be an $n \times n$ matrix that satisfies $P^2 = P$. Show that if $\lambda$ is an eigenvalue of $P$ then $\lambda^2$ is also an eigenvalue of $P$. That this implies the only possible eigenvalues of $P$ are 0 and 1.