1 Partial Derivatives and Total Derivatives

Definition 1.1 For a function \( f : \mathbb{R}^a \to \mathbb{R}^b \), say \( f(x) = (f_1(x), f_2(x), ..., f_b(x)) \), the total derivative \( Df(x, y, z) \) at the point \( x = (x_1, x_2, ..., x_a) \) is a \( b \times a \) matrix whose \( ij \)th entry is

\[
((Df)(x, y, z))_{ij} = \frac{\partial f_i}{\partial x_j}.
\]

For a multivariable function \( f : \mathbb{R}^a \to \mathbb{R}^b \), the total derivative \( Df \) plays the role of the usual derivative in single variable calculus. In s.v.c., the derivative measures the rate of change of the function at a given point. In m.v.c., the total derivative encapsulates the rate of change of \( f \) in each of the linearly independent directions \( e_1, e_2, ..., e_a \) (the standard basis vectors of \( \mathbb{R}^a \)), and in fact it encapsulates how each of the components of \( f = (f_1(x), ..., f_b(x)) \) change in each of those directions. So the entry in the 5th row and 7th column of \( Df \) measures how \( f_5(x_1, ..., x_a) \) is changing in the \( e_7 = (0, 0, 0, 0, 0, 0, 1, 0, 0, ..., 0) \) direction, and this is exactly \( \frac{\partial f_5}{\partial x_7} \).

Exercise 1.2 Autumn 2011 Midterm 2

Let \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) be given by

\[
f(x, y, z) = \begin{pmatrix} x^2 \sin(yz) \\ y^2 + e^{z-x} \end{pmatrix}.
\]

(a) Compute the total derivative \( Df(1, 2, 3) \) at the point \( (1, 2, 3) \).

(b) Compute the total derivative \( Df(x, y, z) \) at a general point \( (x, y, z) \).

Example 1.3 Compute the tangent plane, at an arbitrary point \( (s_0, t_0) \), to the surface in \( \mathbb{R}^3 \) parametrized by

\[
S(s, t) = (e^t \cos s, e^t \sin s, t).
\]
Example 1.4 Consider the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ given by

$$f(x, y, z) = (2x - y, x^2 + y^2 + z^2).$$

Let $a = (1, 1, 1)$. Compute $f(a)$. Now compute the function $\tilde{f}$ obtained by linearly approximating $f(x, y, z)$ at the point $a = (1, 1, 1)$. Hint: recall that in s.v.c., the linear approximation of $f(x)$ at $a$ is given by $\tilde{f} = f(a) + f'(a)(x - a)$.

2 Some More Limits

Exercise 2.1

$$\lim_{(x,y) \to (0,0)} \frac{x}{\sqrt{x^2 + y^2}}.$$ 

Exercise 2.2

$$\lim_{(x,y) \to (0,0)} \frac{\sin((x^2 + y^2)^2)}{(x^2 + y^2)^2}.$$ 

Exercise 2.3

$$\lim_{(x,y) \to (0,0)} \frac{x^2y^5}{2x^4 + 3y^{10}}.$$