1 Review of Determinants

Remark 1.1 In class the determinant was defined as a function from $n \times n$ matrices to real numbers satisfying the following properties

- the determinant is alternating in the rows (exchanging two rows reverses the sign of the determinant)
- the determinant is multilinear with respect to the rows
- the determinant of the identity matrix is 1.

This might not seem like a definition, but it turns out there is a unique function which satisfies these properties, so we call it the determinant. The book instead defines the determinant simply by telling you how to compute it. First there is the formula for $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$ad - bc.$$ 

For a larger matrix, we can compute the determinant recursively by expanding along any row or column.

Remark 1.2 It’s important to that, for computing determinants, rows and columns are treated equally. That is, the determinant is also alternating in the columns and multilinear with respect to the columns!

Question 1.3 How can we view the cross product as a determinant, and can we see why $u \times v$ is perpendicular to $u$ and $v$ using the properties of the determinant?
2 Examples

Example 2.1 Compute the determinant of

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 0 & 2 & 0 \\
0 & 1 & 2 & 3 \\
2 & 3 & 0 & 0
\end{pmatrix}
\]

by expanding along the fourth row, and then expanding along each of the second rows of the resulting $3 \times 3$ matrices.

Example 2.2 Compute the determinant in the previous example by using row and column operations to simplify the computation.

Example 2.3 Is the matrix

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\]

invertible?

3 Exercises

Exercise 3.1 Let $A$ and $B$ be $37 \times 37$ matrices, with $\det(AB) = 5$ and $\det(B) = 3$. Compute $\det(A^{-1})$.

Exercise 3.2 Let $A$ be a $10 \times 10$ matrix. How does the following operations affect $\det A$?

1. exchanging the first two rows of $A$
2. exchanging the first two columns of $A$
3. exchanging the first and third row of $A$
4. exchanging the third and seventh columns of $A$
5. reordering the first three rows of $A$ such that the second row appears first, followed by the third row, followed by the first row, with the remaining rows unchanged
6. multiplying every entry of $A$ by 2
7. replacing the sixth column of $A$ with a column of zeroes
8. performing row operations until $A$ is in reduced row echelon form
9. composing (i.e. multiplying) $A$ with itself

10. inverting $A$

11. adding the row $(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10)$ to $A$ after the 10th row.

12. adding 3 to each element of the fifth column

13. replacing the last row with the sum of all the rows.

**Exercise 3.3**  
(a) Suppose $R$ is the quadrilateral with vertices $(0, 0)$, $(2, 1)$, $(3, -3)$, and $(4, -1)$. Find the area of $R$.

(b) If $T$ is a linear transformation with matrix

$$A = \begin{pmatrix} -3 & 2 \\ -1 & 2 \end{pmatrix}$$

and $R$ is the region from part (a), find the area of the region $T(R)$. 