1 Some Review

*Important:* make sure you understand how to find a basis for the null $N(A)$ space and column space $C(A)$ of a matrix $A$. An important fact is that $N(A) = N(rrefA)$, but $C(A)$ does not always equal $C(rrefA)$! However, computing the reduced row echelon form of $A$ is still very useful for finding a basis of $C(A)$, since recall that we have:

**Proposition 1.1** *Given any matrix $A$, the columns of $R = rref(A)$ which contain the pivots form a basis for $C(R)$, and the corresponding columns of $A$ form a basis for $C(A)$.*

2 Simple Examples of Linear Transformations

A linear transformation $T$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. In other words, it eats vectors in $\mathbb{R}^n$ and spits out vectors in $\mathbb{R}^m$. It needs to satisfy the following two properties, which are what we mean by “linearity”:

- $T(x + y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}^n$
- $T(cx) = cT(x)$.

A basic point is that to each linear transformation we can associate a matrix, and conversely each matrix gives a linear transformation (namely, by multiplying column vectors by that matrix and interpreting the result as the output vector).

Some important examples of linear transformations to understand:

- the identity transformation
- scaling transformations
- diagonal matrices.

In the next lecture, we will introduce two more important examples:

- rotations
Example 2.1 Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ whose matrix is

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$ 

The corresponding formula for $T$ is $T(x_1, x_2) = (x_1, x_1 + x_2)$.

3 Exercises

Exercise 3.1 (Spring 2012 Midterm 1) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^4$ is a linear transformation, and that

$$T \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 4 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad T \left( \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 8 \\ 4 \\ 0 \\ -2 \end{pmatrix}.$$

(a) Find $T \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$ and $T \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$.

(b) Let $b \in \mathbb{R}^4$. Find one or more condition on $b$ that determines precisely whether $b$ is equal to $T(x)$ for some $x \in \mathbb{R}^2$. (Your answer should be given in the form of one or more equations involving the components $b_1, b_2, b_3, b_4$ of $b$.)

Exercise 3.2 (Autumn 2011 Midterm 1)

Let $A$ be a general $m \times n$ matrix. Answer true or false.

1. Multiplying a row by 2 does not change the column space of $A$
2. Multiplying a row by 2 does not change the null space of $A$
3. Multiplying a column by 2 does not change the column space of $A$
4. Multiplying a column by 2 does not change the null space of $A$
5. Multiplying a row by 0 does not change the column space of $A$
6. Multiplying a row by 0 does not change the null space of $A$
7. Switching two rows does not change the null space of $A$
8. Switching two rows does not change the column space of $A$
9. Switching two rows does not change the rank of $A$
10. Switching two columns does not change the nullity of $A$.

**Exercise 3.3** (Spring 2012 Midterm 1) Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$ and let $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2$. Answer true, false, or maybe:

(a) Given a $2 \times 5$ matrix $A$, the equation $Ay = e_1$ has no solutions $y$ in $\mathbb{R}^5$.

(b) Given a $5 \times 2$ matrix $B$, the equation $Bz = Be_1$ has infinitely many solutions $z$ in $\mathbb{R}^2$.

(c) Given vectors $v_1, v_2, v_3$ in $\mathbb{R}^2$ with the property that each of the sets

\[
\{v_1, v_2\}, \quad \{v_2, v_3\}, \quad \text{and} \quad \{v_1, v_3\}
\]

is linearly independent, the set $\{v_1, v_2, v_3\}$ is also linearly independent.

(d) Given vectors $w_1, w_2, w_3$ in $\mathbb{R}^5$ with the property that each of the sets

\[
\{w_1, w_2\}, \quad \{w_2, w_3\}, \quad \text{and} \quad \{w_1, w_3\}
\]

is linearly independent, the set $\{w_1, w_2, w_3\}$ is also linearly independent.

(e) Given nonzero $v \in \mathbb{R}^2$, the set

\[
V = \{x \in \mathbb{R}^2 \mid ||x + v||^2 = ||x||^2 + ||v||^2\}
\]

is a subspace of $\mathbb{R}^2$.

(f) Given nonzero $w \in \mathbb{R}^2$, the set

\[
W = \{x \in \mathbb{R}^2 \mid ||x + w|| = ||x|| + ||w||\}
\]

is a subspace of $\mathbb{R}^2$. 
