1 Review from lecture

- linear combinations
- span
- linear independence

**Example 1.1** Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$. Examples of linear combinations of $v_1$ and $v_2$ include

- $1 \cdot v_1 + 1 \cdot v_2 = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$
- $5 \cdot v_1 + 2 \cdot v_2 = \begin{pmatrix} 9 \\ 10 \\ 13 \end{pmatrix}$
- $\pi \cdot v_1 + 0 \cdot v_2 = \begin{pmatrix} \pi \\ 2\pi \\ 3\pi \end{pmatrix}$.

**Exercise 1.2** In the above example, what is an example of a vector in $\mathbb{R}^3$ which is not a linear combination of $v_1$ and $v_2$?

In $\mathbb{R}^3$ there is the “standard basis” $e_1, e_2, e_3$, where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
Proposition 1.3  The set \{e_1, e_2, e_3\} is linearly independent.

Proof  Suppose \(c_1e_1 + c_2e_2 + c_3e_3 = 0\) for \(c_1, c_2, c_3 \in \mathbb{R}\). This means that

\[
\begin{pmatrix}
  c_1 \\
  c_2 \\
  c_3
\end{pmatrix} = \begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix}.
\]

This is equivalent to the three equations: \(c_1 = 0\), \(c_2 = 0\), and \(c_3 = 0\), and hence we have linear independence.

2  Parametric equation of a plane

We want to find a “parametric equation” for a plane in \(\mathbb{R}^3\), similar to what to found for lines before. If we have vectors \(v_1, v_2 \in \mathbb{R}^3\), the plane “spanned by” \(v_1\) and \(v_2\), i.e. the plane containing \(v_1, v_2\), and the origin, is of the form

\[
\{sv_1 + tv_2 \mid s, t \in \mathbb{R}\}.
\]

If we instead we want the plane which is parallel to the above one but passes through \(w\) instead of \(0\), we get

\[
\{w + sv_1 + tv_2 \mid s, t \in \mathbb{R}\}.
\]

If we have vectors \(v_1, v_2, v_3 \in \mathbb{R}^3\), suppose we want the plane which passes through \(v_1, v_2, v_3\). Then we can take

\[
\{v_1 + s(v_2 - v_1) + t(v_3 - v_1) \mid s, t \in \mathbb{R}\}.
\]

Exercise 2.1  What is a parametric equation of the plane in \(\mathbb{R}^3\) passing through \(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\), \(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\), and \(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\)? Try to see why your answer makes geometric sense.

3  Exercises

Exercise 3.1  Let \(\{u, v, w\}\) be a linearly independent set. Is \(\{u - v, v - w, u - w\}\) a linearly independent set? Show that it is or show why it is not.
3.1 True / false

Exercise 3.2 If $S = \{v_1, ..., v_k\}$ is a set of linearly independent vectors in $\mathbb{R}^n$, then any subset of $S$ must be linearly independent.

Exercise 3.3 If $S = \{v_1, ..., v_k\}$ is a set of linearly dependent vectors in $\mathbb{R}^n$, then any subset of $S$ must be linearly dependent.

Exercise 3.4 If $\text{span}(v_1, v_2, v_3) = \mathbb{R}^3$, then $\{v_1, v_2, v_3\}$ must be a linearly independent set.

Exercise 3.5 If $S = \{v_1, v_2, v_3\}$ is a linearly dependent set, then every vector in $S$ can be written as a linear combination of the other 2 vectors.